Commutative algebra

Contents

1 Primary Decomposition

1.1 Prime Avoidance

Lemma 1.1 (Prime avoidance)**.** Suppose that $I_1, \ldots, I_n$, J are ideals of a ring A, and suppose that $J \subseteq \bigcup_{j=1}^{n} I_j$. If A contains an infinite field or if at most two of the I_j are not prime, then J is contained in one of the I_j.

Proof. First assume that A contains an infinite field k. Then this follows from the claim below.

Lemma 1.2. Let k be a field, V be a k-vector space and suppose V is the union of its proper subspace $W_1, \ldots, W_n$. Then $\#k \leqslant n-1$.

Proof. We may assume no W_i is contained in the union of the other subspaces. Let $u \in W_i \backslash \bigcup_{j \neq i} W_j$ and $v \notin W_i$. Then

$$(v + ku) \cap W_i = \varnothing$$

and $(v + Fu) \cap W_j$ contains at most one vector, for otherwise W_j would contain u. Hence

$$\#(v + ku) = \#k \leqslant n - 1$$

$\square$

Now suppose the latter condition, and assume each I_j is not contained in the other ideals. We prove it by induction on n, $n = 1$ being trivial. Suppose $n = 2$ and $J \subseteq I_1 \cup I_2$. If $J \nsubseteq I_1$ and $J \nsubseteq I_2$, pick $s \in J - I_1$ and $x \in J - I_2$, then $x + s \notin I_2$. Hence x and $x + s$ lie in I_1, so that $s \in I_1$, a contradiction.

When $n > 2$, assume that I_n is a prime. Then $JI_1 \cdots I_{n-1} \nsubseteq I_n$; take $x \in JI_1 \cdots J_{n-1} - I_n$. Suppose $S = J - (I_1 \cup \cdots \cup I_{n-1})$; by induction $S \neq \varnothing$. Since $J \subseteq I_1 \cup \cdots \cup I_n$, S is contained in I_n. But if $s \in S$, then $s + x \in S$, and hence both s and $s + x$ are in I_n, implying $x \in I_n$, a contradiction. $\square$

Remark 1.3. In the case not involving a ground field, the proof above only use that J is a subring of R without unit.

Lemma 1.4. Let A be a graded ring and $J \subseteq A_+$ be an homogeneous ideal. If $I_1, \ldots, I_n$ are prime ideals of A such that all homogeneous elements of J are contained in $I_1 \cup \cdots \cup I_n$, then J is contained in some I_j.

Proof. The proof is almost the same as above. Assume each I_j is not contained in the other ideals. We use induction on n, $n = 1$ being trivial. Suppose $n = 2$ and $J \subseteq I_1 \cup I_2$. Assume otherwise that there exist homogeneous $s \in J - I_1$ and $x \in J - I_2$. Lifting to large powers and keeping I_1, I_2 are primes in mind, we may assume x and s are of the same degree. Then $x + s \notin I_2$, which implies x and $x + s$ lie in I_1. Hence $s \in I_1$, a contradiction.

For $n > 2$, suppose otherwise. Then $JI_1 \cdots I_{n-1} \nsubseteq I_n$; take homogeneous $x \in JI_1 \cdots J_{n-1} - I_n$. Suppose $S = J - (I_1 \cup \cdots \cup I_{n-1})$; by induction S contains a homogeneous element. Since $J \subseteq I_1 \cup \cdots \cup I_n$, S is contained in I_n. Take a homogeneous $s \in S$ and raise x, s to a power so that they have the same degree. But then $s + x \in S$, and hence both s and $s + x$ are in I_n, implying $x \in I_n$, a contradiction. $\square$

1.2 Associated Primes

Let A be a ring and M be an A-module.

Lemma 1.5. Let S a multiplicatively closed set of A, and assume that $0 \notin S$. Then there exists and ideal of A which is maximal in the set of ideal not intersecting S, and any such ideal is prime.

Proof. The existence of such an ideal $\mathfrak{p}$ results from Zorn's lemma. Let $\mathfrak{p}$ be such a maximal ideal. Let $a, b \in A$ with $ab \in \mathfrak{p}$ and $a, b \notin \mathfrak{p}$. Then $(a, \mathfrak{p})$ and $(b, \mathfrak{p})$ meet S, so there exist $s, s' \in S$ with $s \in (a, \mathfrak{p})$, $s' \in (b, \mathfrak{p})$. Then $S \ni ss' \in (a, \mathfrak{p})(b, \mathfrak{p}) \subseteq \mathfrak{p}$, a contradiction. $\square$

Corollary 1.5.1. The nilradical of A is the intersection of all prime ideals in A.

Corollary 1.5.2. The radical of an ideal in A is the intersection of all prime ideals containing I.

Definition. For a submodule N of M, the ideal $\operatorname{ann}_A(N) := \{a \in A \mid aN = 0\}$ is called the **annihilator** of N. For $x \in M$, the **annihilator** of x is the ideal $\operatorname{ann}_A(x) = \operatorname{ann}_A(xA)$.

- By the first isomorphism theorem, we have

$$\begin{aligned} A/\operatorname{ann}_A(x) &\xrightarrow{\sim} Ax \\ a + \operatorname{ann}_A(x) &\longmapsto ax \end{aligned}$$

- For a prime ideal $\mathfrak{p}$, $(Ax)_{\mathfrak{p}} \neq 0$ if and only if $\operatorname{ann}_A(x) \subseteq \mathfrak{p}$.

  Proof. By the isomorphism above, $(Ax)_{\mathfrak{p}} \neq 0$ iff $\operatorname{ann}_A(x)_{\mathfrak{p}} \neq A_{\mathfrak{p}}$, iff $\operatorname{ann}_A(x) \subseteq \mathfrak{p}$. $\square$

For $a \in A$, denote by $a_M \in \operatorname{End}_A(M)$ the homomorphism $x \mapsto ax$. a_M is called **locally nilpotent** if for each $x \in M$, $a^n x = 0$ for $n \gg 1$.

- If M is a finite A-module, then a_M is locally nilpotent if and only if a_M is nilpotent (as an element of $\operatorname{End}_A(M)$).

The **support** of M is the set

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\}$$

- If M is a finite A-module, then $\operatorname{supp}(M) = V(\operatorname{ann}_A(M))$, so that $\operatorname{supp}(M)$ is closed in $\operatorname{Spec}(A)$. Even if M is not finite over A, we still have $\operatorname{supp}(M) \subseteq V(\operatorname{ann}_A(M))$.

Proof. Say $M = Ax_1 + \cdots + Ax_n$. We have $M_{\mathfrak{p}} \neq 0$ iff $(Ax_i)_{\mathfrak{p}} \neq 0$ for some $i = 1, \ldots, n$, iff $\operatorname{ann}_A(x_i) \subseteq \mathfrak{p}$, i.e. $\mathfrak{p} \in \bigcup_{i=1}^{n} V(\operatorname{ann}_A(x_i)) = V(\operatorname{ann}_A(M))$ $\square$

Proposition 1.6. Let $a \in A$. Then a_M is locally nilpotent if and only if $a \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{supp}(M)$.

Proof. Suppose a_M is locally nilpotent and let $\mathfrak{p} \in \operatorname{supp}(M)$. Then there exists $x \in M$ with $(Ax)_{\mathfrak{p}} \neq 0$; let $n \in \mathbb{N}$ such that $a^n x = 0$. Then $a^n \in \operatorname{ann}_A(x) \subseteq \mathfrak{p}$, as wanted.

Conversely, suppose a_M is not locally nilpotent, so that there exists $x \in M$ such that $a^n x \neq 0$ for all $n \geqslant 0$. Apply Lemma 1.5 to $S = \{1, a, a^2, \ldots\}$ to obtain a prime $\mathfrak{p}$ of A outside S. Then $a \notin \mathfrak{p}$ and $(Ax)_{\mathfrak{p}} \neq 0$, implying $\mathfrak{p} \in \operatorname{supp}(M)$. $\square$

Definition. A prime $\mathfrak{p}$ is **associated** to M if there exists $x \in M$ such that $\mathfrak{p} = \operatorname{ann}_A(x)$. The set of associated primes M is denoted by $\operatorname{Ass}_A(M)$.

- For a prime $\mathfrak{p}$, $\mathfrak{p} \in \operatorname{Ass}_A(M)$ iff there is an injective A-module homomorphism $A/\mathfrak{p} \to M$.
- If $\mathfrak{p} = \operatorname{ann}_A(x)$ for some $x \in M$, since $\mathfrak{p} \neq A$, we have $x \neq 0$. Hence, if $M = 0$, then $\operatorname{Ass}_A(x) = \varnothing$.
- For $M \neq 0$, the maximal element $\mathfrak{p}$ among the set of ideals $\{\operatorname{ann}_A(x) \mid x \in M - \{0\}\}$ is prime.

  Proof. Let $\mathfrak{p}$ be such in the statement and $\mathfrak{p} = \operatorname{ann}_A(x)$ for some $x \in M$. Let $a, b \in A$ with $ab \in \mathfrak{p}$, $a \notin \mathfrak{p}$; then $ax \neq 0$. By maximality, $\operatorname{ann}_A(ax) = \operatorname{ann}_A(x)$, and since $bax = 0$, $b \in \operatorname{ann}_A(ax) = \operatorname{ann}_A(x) = \mathfrak{p}$. $\square$

- In particular, if A is Noetherian and $M \neq 0$, then $\operatorname{Ass}_A(M) \neq \varnothing$.

Proposition 1.7. Assume A, M are Noetherian and $M \neq 0$. Then there exists a chain of submodules

$$M = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_r = 0$$

with each factor $M_i/M_{i+1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$.

Proof. Consider the set of submodules of M having the property described above; it is nonempty since for $\mathfrak{p} = \operatorname{ann}_A(x) \in \operatorname{Ass}_A(M)$, $Ax \cong A/\mathfrak{p}$. Then it has a maximal element, say N. If $N \neq M$, then $M/N \neq 0$, and we can pick $\mathfrak{q} = \operatorname{ann}_A(x + N) \in \operatorname{Ass}_A(M/N)$ with some $x \in M - N$; in particular, $xA + N/N \cong A/\mathfrak{q}$, and this contradicts to the maximality of N. Hence $M = N$. $\square$

Proposition 1.8. Let A be Noetherian and $a \in A$. Then a_M is injective if and only if a does not lie in any associated prime of M.

Proof. Suppose a_M is injective. Then a cannot annihilate any element of M. Conversely, suppose a_M is not injective; say $ax = 0$ for some $x \neq 0$. Then $Ax \neq 0$, so that $\operatorname{Ass}_A(Ax) \neq 0$, and thus a lies in an associated prime of Ax, hence of M. $\square$

Proposition 1.9. Let A be Noetherian, and let M be a module. Let $a \in A$. TFAE:

(i) a_M is locally nilpotent.

(ii) a lies in every associated prime of M.

(iii) a lies in every prime $\mathfrak{p} \in \operatorname{supp}(M)$

If $\mathfrak{p} \in \operatorname{supp}(M)$, then $\mathfrak{p}$ contains an associated prime of M.

Proof. It remains to show (ii)$\Rightarrow$ (iii), which is implied by the last statement. Now let $\mathfrak{p} \in \operatorname{supp}(M)$. Then there exists $x \in M$ with $(Ax)_{\mathfrak{p}} \neq 0$, and there exists an associated prime $\mathfrak{q}$ of $(Ax)_{\mathfrak{p}}$; say $\mathfrak{q} = \operatorname{ann}_A(r/s)$, where $r \in Ax$ and $s \notin \mathfrak{p}$. Then $\mathfrak{q} \subseteq \mathfrak{p}$, for otherwise there exists $b \notin \mathfrak{q} - \mathfrak{p}$ (so that b is invertible in $A_{\mathfrak{p}}$) with $b(y/s) = 0$, hence $y/s = 0$, a contradiction. Finally, let $\mathfrak{q} = (b_1, \dots, b_n)$. Since $b_i \in \operatorname{ann}_A(r/s)$, there exist $s_1, \dots, s_n \notin \mathfrak{p}$ with $b_i s_i r = 0$. Put $t = s_1 \cdots s_n$. Then $\mathfrak{q} = \operatorname{ann}_A(tr)$ is an associated prime of M. □

Corollary 1.9.1. Let A be a Noetherian ring and M a finite A-module. Then

$$\sqrt{\operatorname{ann}_A(M)} = \bigcap_{\mathfrak{p} \in \operatorname{supp}(M)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$$

Corollary 1.9.2. Let A be a Noetherian ring and M be an A-module. TFAE:

(i) $\# \operatorname{Ass}(M) = 1$.

(ii) $M \neq 0$, and for every $a \in A$, a_M is either injective or locally nilpotent.

In these conditions are satisfied, then the set $\{a \in A \mid a_M \text{ is locally nilpotent}\}$ equals the associated prime.

Proof. (i)$\Rightarrow$(ii) and the last assertion are clear from the above propositions. Suppose that (ii) holds, and suppose that there are two distinct associated primes, say $\operatorname{ann}_A(x)$ and $\operatorname{ann}_A(y)$ for some $x, y \in M$. WLOG, we assume $\operatorname{ann}_A(x) - \operatorname{ann}_A(y) \neq \varnothing$, and pick an element a in it. In particular, a is not injective, so that a is locally nilpotent, which implies that $a \in \operatorname{ann}_A(y)$, a contradiction. □

Proposition 1.10. Let N be a submodule of M. Every associated prime of N is associated with M. An associated prime of M is associated with either N or with M/N.

Proof. The first assertion is clear. Now let $\mathfrak{p} = \operatorname{ann}_A(x)$ be an associated prime of M. If $Ax \cap N = 0$, then Ax is isomorphic to a submodule of M/N, so that $\mathfrak{p} \in \operatorname{Ass}(M/N)$. Suppose now that $Ax \cap N \neq 0$, say $y := ax \in N - \{0\}$. Clearly, $\operatorname{ann}_A(y) \supseteq \operatorname{ann}_A(x)$. Now let $b \in A$ with $by = 0$, i.e. $bax = 0$. Then $ba \in \operatorname{ann}_A(x)$. Since $ax \neq 0$ and $\operatorname{ann}_A(x)$ is a prime, $b \in \operatorname{ann}_A(x)$ so that $\operatorname{ann}_A(y) = \operatorname{ann}_A(x) = \mathfrak{p} \in \operatorname{Ass}(N)$. □

Corollary 1.10.1. Assume A and M are Noetherian. Then $\operatorname{Ass}(M)$ is finite.

Proof. Using Proposition above and Proposition 1.7, we have

$$\operatorname{Ass}(M) \subseteq \bigcup_{i=1}^{r-1} \operatorname{Ass}(M_i/M_{i+1}) = \bigcup_{i=1}^{r-1} \operatorname{Ass}(A/\mathfrak{p}_i)$$

with $\mathfrak{p}_i \in \operatorname{Spec}(A)$. Generally, for $\mathfrak{p} \in \operatorname{Spec}(A)$ and $x \in A - \mathfrak{p}$,

$$\operatorname{ann}_A(x+\mathfrak{p}) = \{a \in A \mid a(x+\mathfrak{p}) = \mathfrak{p}\} = \{a \in A \mid ax \in \mathfrak{p}\} = \mathfrak{p}$$

so that $\# \operatorname{Ass}(A/\mathfrak{p}) = 1$. Hence $\operatorname{Ass}(M)$ is finite. □

Proposition 1.11. Let A be a Noetherian ring and $M \neq 0$ an A-module. Then for any m.c.s. S of A,

$$\operatorname{Ass}_{S^{-1}A}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_A(M),\, \mathfrak{p} \cap S = \varnothing\} = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_A(S^{-1}M)\}$$

Proof. Denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively, the three sets from left to right.

- $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathfrak{p} \in \operatorname{Ass}(M)$. Then $A/\mathfrak{p} \subseteq M$, so that $S^{-1}A/S^{-1}\mathfrak{p} \subseteq S^{-1}M$. Also $S^{-1}\mathfrak{p}$ is a prime ideal iff $\mathfrak{p} \cap S = \varnothing$, and if it is the case, we obtain $S^{-1}\mathfrak{p} \in \operatorname{Ass}_{S^{-1}A}(S^{-1}M)$.

- $\mathcal{A} \subseteq \mathcal{C}$. Suppose $\mathfrak{q} \in \operatorname{Ass}_{S^{-1}A}(S^{-1}M)$. Then there exists $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\mathfrak{q} = S^{-1}\mathfrak{p}$. Since $\mathfrak{q}$ is associated to $S^{-1}M$, say $\mathfrak{q} = \operatorname{ann}_{S^{-1}A}(x)$ for some $x \in S^{-1}M$, we have $\mathfrak{p} = \operatorname{ann}_A(x)$.

- $\mathcal{C} \subseteq \mathcal{B}$. Let $\mathfrak{p} = \operatorname{ann}_A(m/s) \in \operatorname{Ass}_A(S^{-1}M)$ with $m \in M$, $s \in S$, $m/s \neq 0$. If $\mathfrak{p} \cap S \neq \varnothing$, say r belongs to the intersection, then $rm/s = 0$. Then $m/s = rm/rs = 0$, a contradiction. Clearly, $\operatorname{ann}_A(m) \subseteq \mathfrak{p}$; if it's not an equality, pick $b \in \mathfrak{p} - \operatorname{ann}_A(m)$. Then $bms' = 0$ for some $s' \in S$, so that

  $$\operatorname{ann}_A(m) \subsetneq \operatorname{ann}_A(s'm) \subseteq \mathfrak{p}$$

  Repeating in this way, since A is Noetherian, it must be the case $\mathfrak{p} = \operatorname{ann}_A(s''m)$ for some $s'' \in S$.

- $\mathcal{A} \subseteq \mathcal{B}$. Suppose $\mathfrak{q} \in \operatorname{Spec}(S^{-1}A)$ that is associated to $S^{-1}M$; then $\mathfrak{q} = S^{-1}\mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \cap S = \varnothing$. Then there is an injection $\varphi : S^{-1}A/S^{-1}\mathfrak{p} \to S^{-1}M$. Since $\mathfrak{p}$ is finitely generated, we have

  $$\operatorname{Hom}_{S^{-1}A}(S^{-1}A/S^{-1}\mathfrak{p}, S^{-1}M) \cong S^{-1} \operatorname{Hom}_A(A/\mathfrak{p}, M)$$

  so we can write $\varphi = f/s$ for some $f : A/\mathfrak{p} \to M$ and $s \in S$. Since $S \cap \mathfrak{p} = \varnothing$, u is a nonzerodivisor on $A/\mathfrak{p}$, so that $f = s\varphi$ is injective, showing that $A/\mathfrak{p}$ is isomorphic to a submodule of M.

□

Corollary 1.11.1. Let A be a Noetherian ring and $M \neq 0$ a finite A-module. Then $\operatorname{Ass}_A(M)$ includes all primes minimal among the primes containing $\operatorname{ann}_A(M)$.

Proof. Let $\mathfrak{p}$ be the prime minimal over $\operatorname{ann}_A(M)$; in particular, $M_{\mathfrak{p}} \neq 0$.

Method I Then $\mathrm{Ass}_A(M_\mathfrak{p}) \neq \varnothing$, say $\mathfrak{q} \in \mathrm{Ass}_A(M_\mathfrak{p})$. By Proposition, $\mathfrak{q} \in \mathrm{Ass}_A(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$, so $\mathfrak{q} \supseteq \mathrm{ann}_A(M)$. Since $\mathfrak{p}$ is minimal, $\mathfrak{p} = \mathfrak{q}$, as shown.

Method II Since M is finite over A, $S^{-1}\mathrm{ann}_A(M) = \mathrm{ann}_{S^{-1}A}(S^{-1}M)$. Localizing at $\mathfrak{p}$, we can assume $(A, \mathfrak{p})$ is a local ring. Then $\mathfrak{p}$ is the only prime containing $\mathrm{ann}_A(M)$, so $\mathfrak{p} \in \mathrm{Ass}_A(M)$, as shown in Method I. Finally use Proposition to validate that we can take localization. □

1.3 Primary Decomposition

Let A be a ring and M be an A-module.

Definition. A submodule Q of M is **primary** if $Q \neq M$ and for every $a \in A$, $a_{M/Q}$ is either injective or nilpotent.

- Viewing A itself as an A-module, an ideal $\mathfrak{q}$ is **primary** iff for $a, b \in A$ with $ab \in \mathfrak{q}$ and $a \notin \mathfrak{q}$, we have $b^n \in \mathfrak{q}$ for some $n \geqslant 1$, namely, every nonzerodivisor of $A/\mathfrak{q}$ is nilpotent.

  Proof. Let $a, b \in A$ with $ab \in \mathfrak{q}$, $a \notin \mathfrak{q}$. Then $b_{A/\mathfrak{q}}$ is not injective, so $0 = b^n_{A/\mathfrak{q}}(1 + \mathfrak{q})$ for some n, i.e. $b^n \in \mathfrak{q}$. Conversely, let $a \in A$ with $a_{A/\mathfrak{q}}$ is not injective; let $b \in A - \mathfrak{q}$ such that $ab \in \mathfrak{q}$. Then $a^n \in \mathfrak{q}$ for some $n \in \mathbb{N}$, showing that $a^n_{A/\mathfrak{q}} = 0$, i.e. $a_{A/\mathfrak{q}}$ is nilpotent.

  The essence is that $1 \in A$, so being nilpotent is equivalent to being locally nilpotent. □

- For Q primary, let $\mathfrak{p} = \{a \in A \mid a_{M/Q} \text{ is nilpotent}\} = \sqrt{\mathrm{ann}_A(M/Q)}$. Then $\mathfrak{p}$ is a prime, and we say Q is **$\mathfrak{p}$-primary**, or $\mathfrak{p}$ belongs to Q. In particular, if $\mathfrak{q}$ is a primary ideal, then $\mathfrak{p} = \sqrt{\mathfrak{q}}$.

  Proof. Let $a, b \in A$ with $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$. Then $a_{M/Q}$ is injective, so b is nilpotent, showing $b \in \mathfrak{p}$. □

- For $a \in A$, $x \in M$ with $ax \in Q$, if $x \notin Q$, then $a \in \mathfrak{p}$.

- If $\mathfrak{m} \in \mathrm{mSpec}(A)$ and $\mathfrak{q}$ is an ideal of A with $\mathfrak{m}^n \subseteq \mathfrak{q}$ for some $n \geqslant 1$, then $\mathfrak{q}$ is $\mathfrak{m}$-primary.

  Proof. Taking radical, we have $\mathfrak{m} \subseteq \sqrt{\mathfrak{q}}$. Since $\mathfrak{m}$ is maximal, we have $\mathfrak{m} = \sqrt{\mathfrak{q}}$, and this implies $\mathfrak{m}$ is the only prime containing $\mathfrak{q}$ i.e. $A/\mathfrak{q}$ has only one prime ideal $\mathfrak{m}$. This shows every element in $A/\mathfrak{q}$ is either a unit or nilpotent. □

- If $Q_1, \ldots, Q_r$ are all $\mathfrak{p}$-primary submodules of M, then so is $Q := Q_1 \cap \cdots \cap Q_r$.

  Proof. Let $a \in \mathfrak{p}$ and $n_1, \ldots, n_r$ be positive integers such that $a^{n_i}_{M/Q_i} = 0$ for $i = 1, \ldots, r$. Then $a^n_{M/Q} = 0$ for $n \geqslant \max\{n_1, \ldots, n_r\}$, so that $a_{M/Q}$ is nilpotent. Now if $a \notin \mathfrak{p}$, then $a^n x \notin Q_i$ for $x \in M - Q_i$, $n \geqslant 1$ and all i; hence $a^n x \notin Q$ for all $x \in M - Q$, so that $a_{M/Q}$ is injective. □

Let N be a submodule of M. A **primary decomposition** of N is a finite collection of primary submodules $Q_1, \ldots, Q_r$ of M with

$$N = Q_1 \cap \cdots \cap Q_r$$

- Using the property above, after grouping we can assume each Q_i is $\mathfrak{p}_i$-primary with each $\mathfrak{p}_i$ distinct.
- If each $\mathfrak{p}_i$ is distinct, and $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$ for each i, we say it's a **reduced/irredundant/minimal** primary decomposition. If N admits primary decomposition, then it also admits an irredundant one.
- For $Q, N \leqslant M$, Q is primary if and only if Q mod N is primary, and the prime belonging to them is the same.
- N has a primary decomposition in M if and only if (0) has a primary decomposition in M/N. Furthermore, if $N = Q_1 \cap \cdots \cap Q_r$, then $(0) = \overline{Q_1} \cap \cdots \cap \overline{Q_r}$, where $\overline{Q_i}$ denotes the image of Q_i in M/N; the decomposition of N is irredundant iff that of (0) is irredundant.

Let $N = Q_1 \cap \cdots \cap Q_r$ be an irredundant primary decomposition, and let $\mathfrak{p}_i$ belong to Q_i. An **isolated prime** is a minimal element in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Theorem 1.12. Let N be a submodule of M, and let

$$N = Q_1 \cap \cdots \cap Q_r = Q'_1 \cap \cdots \cap Q'_s$$

be irredundant primary decompositions. Then

(i) $r = s$.

(ii) The set of primes belonging to $Q_1, \ldots, Q_r$ and $Q'_1, \ldots, Q'_s$ is the same.

(iii) If $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ is the set of isolated primes, then $Q_i = Q'_i$ for $i = 1, \ldots, m$. In other words, the primary modules corresponding to isolated primes are uniquely determined.

Theorem 1.13. Let M be Noetherian, and $N \leqslant M$. Then N admits a primary decomposition.

Proposition 1.14. Let A, M be Noetherian. A submodule $Q \leqslant M$ is primary if and only if $\mathrm{Ass}_A(M/Q) = \{\mathfrak{p}\}$ is a singleton, and in this case, Q is $\mathfrak{p}$-primary.

Theorem 1.15. Let A, M be Noetherian. The associated primes of M are precisely the primes which belongs to the primary modules in a reduced primary decomposition of 0 in M. This in particular again shows that $\mathrm{Ass}_A(M)$ is finite.

1.4 Factoriality

Lemma 1.16. For a Noetherian domain A, if every irreducible element in A is prime, then A is a UFD.

Proposition 1.17. Let A be a Noetherian domain.

1. If $f \in A$ and $f = up_1^{e_1} \cdots p_r^{e_r}$, where $u \in A^\times$, the p_i are primes in A generating distinct prime ideals $(p_i) \in \operatorname{Spec}(A)$ and $e_i \in \mathbb{N}$, then
$$f = (p_1^{e_1}) \cap \cdots \cap (p_r^{e_r})$$
are the minimal primary decomposition of the ideal (f).

2. R is a UFD if and only if every prime ideal minimal over a principal ideal is itself principal.

2 Graded rings

In this section, the words “ideal” and ”proper ideal” are interchangeable.

Definition. A **graded ring** is a ring A together with a family of subgroups $\{A_n\}_{n\geqslant 0}$ of the additive group A such that

(i) $A = \bigoplus_{n\geqslant 0} A_n$ as abelian groups;

(ii) $A_nA_m \subseteq A_{n+m}$ for all $n, m \geqslant 0$.

- By definition A_0 is a subring of A, and each A_n is an A_0-module.
- The subgroup $A_+ := \bigoplus_{n\geqslant 1} A_n$ is an ideal of A.

A **graded A-module** M is an A-module together with a family of subgroups $\{M_n\}_{n\geqslant 0}$ of M such that

(i) $M = \bigoplus_{n\geqslant 0} M_n$ as abelian groups;

(ii) $A_nM_m \subseteq M_{n+m}$ for all $n, m \geqslant 0$.

- Each M_n is an A_0-module.

An element $x \in M$ is **homogeneous** if $x \in M_n$ for some n, and n is called the **degree** of x. For each element $y \in M$, $y = y_1 + \cdots + y_r$ for unique homogeneous elements $y_1, \ldots, y_r \in M$ of distinct degrees. These y_i are called the **homogeneous components** of y.

Definition. Let A be a graded ring and M, N be graded A-module. A **homomorphisms of graded A-modules** from M to N is a A-module homomorphism $f : M \to N$ with $f(M_n) \subseteq N_n$ for each $n \geqslant 0$.

Proposition 2.1. Let A be a graded ring. Then A is Noetherian if and only if A_0 is Noetherian and A is a finitely generated A_0-algebra.

Proof. The only if part follows from Hilbert’s basis theorem. For the if part, immediately we see $A_0 = A/A_+$ is Noetherian. Since A_+ is an proper ideal of A, $A_+ = Ax_1 + \cdots + Ax_r$ for some $x_i \in A_+$; we may assume that each x_i is homogeneous. Let $A' = A_0[x_1, \ldots, x_r]$

Now we prove by induction on $n \geqslant 0$ that $A_n \subseteq A'$, the case $n = 0$ being trivial. For $n > 0$, let $x \in A_n$. Then $x = a_1x_1 + \cdots a_rx_r$ for some $a_r \in A$.

- We may assume each a_i is homogeneous. Write $a_i = a_{i1} + \cdots + a_{ir_i}$ with each a_{ij} homogeneous of distinct degrees. Since $x \in A_n$, x equals the sum of those $a_{ij}x_i$ with degree $\deg a_{ij} + \deg x_i = n$, and those with degree $\neq n$ may offset.

- Since each x_i has positive degree, each a_i is of degree strictly smaller than n. Hence by induction hypothesis we $a_i \in A'$, and thus $x \in A'$, as wanted.

$\square$

Let A be a ring, and $\mathfrak{a}$ an ideal of A. We can form a graded ring

$$\mathrm{Bl}_{\mathfrak{a}} A = A^* := \bigoplus_{n \geqslant 0} \mathfrak{a}^n$$

called the **blowup algebra** of $\mathfrak{a}$ in A.

- If A is Noetherian, then $\mathfrak{a}$ is finitely generated A-module, and hence A^* is Noetherian by the previous theorem.

Let M be an A-module. A chain of submodules of M

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is called an **$\mathfrak{a}$-filtration** of M if $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n. Then

$$M^* = \bigoplus_{n \geqslant 0} M_n$$

is a graded A^*-module, since $\mathfrak{a}^m M_n \subseteq M_{m+n}$.

Lemma 2.2. Let A be Noetherian, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module, and $\{M_n\}_{n \geqslant 0}$ an $\mathfrak{a}$-filtration. TFAE:

1. M^* is a finitely generated A^*-module.
2. The filtration $\{M_n\}_{n \geqslant 0}$ is **stable**, i.e., $\mathfrak{a}M_n = M_{n+1}$ for $n \gg 0$.

Proof. Since M is Noetherian, each M_n is finitely generated, and hence so is each $Q_n = \bigoplus_{r=0}^{n} M_r$. Then Q_n generates an A^*-submodule of M^*, namely

$$M_n^* = M_0 \oplus \cdots \oplus M_n \oplus \mathfrak{a}M_n \oplus \cdots \oplus \mathfrak{a}^r M_n \oplus \cdots$$

Since Q_n is finitely generated as an A-module, M_n^* is finitely generated as an A^*-module. The submodules M_n^* forms an ascending chain. Now, since A^* is Noetherian, we see

$$M^* \text{ is finitely generated as an } A^*\text{-module} \Leftrightarrow \text{the chain } \{M_n^*\} \text{ stops} \Leftrightarrow M_n^* = M^* \text{ for some } n \geqslant 0.$$

and it is equivalent to saying that $M_{n+r} = \mathfrak{a}^r M_n$ for all $r \geqslant 0$, i.e., the filtration $\{M_n\}_{n \geqslant 0}$ is stable. $\square$

2.1 Artin-Rees Lemma

Proposition 2.3 (Artin-Rees)**.** Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module, $\{M_n\}_{n\geqslant 0}$ a stable $\mathfrak{a}$-filtration of M. If M' is a submodule of M, then $\{M' \cap M_n\}_{n\geqslant 0}$ is a stable $\mathfrak{a}$-filtration of M'.

Proof.

- Since $\{M_n\}$ is stable, Lemma shows that M^* is a finitely generated A^*-module. Since A^* is Noetherian, M^* is Noetherian.
- $\{M' \cap M_n\}_{n\geqslant 0}$ is an $\mathfrak{a}$-filtration of M'. Indeed, $\mathfrak{a}(M' \cap M_n) \subseteq M' \cap \mathfrak{a}M_n \subseteq M' \cap M_{n+1}$.
- $M'^* := \bigoplus_{n\geqslant 0}(M' \cap M_n)$ is a graded A^*-submodule of M^*, so M'^* is finitely generated. By Lemma again, $\{M' \cap M_n\}_{n\geqslant 0}$ is stable.

$\square$

Specializing to the filtration $\{\mathfrak{a}^n M\}_{n\geqslant 0}$, we see

Corollary 2.3.1. There exists $k \in \mathbb{N}$ such that

$$\mathfrak{a}^n M \cap M' = \mathfrak{a}^{n-k}((\mathfrak{a}^k M) \cap M')$$

for all $n \geqslant k$.

2.2 Krull Intersection Theorem

Theorem 2.4 (Krull Intersection Theorem)**.** Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, M a finitely generated A-module. Then $\bigcap_{n\geqslant 1} \mathfrak{a}^n M$ consists of those $x \in M$ annihilated by $1 - a$ for some $a \in \mathfrak{a}$.

Proof. Put $E = \bigcap_{n\geqslant 1} \mathfrak{a}^n M$. By Corollary 2.3.1, we see $\mathfrak{a}E = E$ by taking $n = k + 1$. Since M is finitely generated and A is Noetherian, E is finitely generated. Using a determinant argument, from $\mathfrak{a}E = E$ we deduce that $(1 - a)E = 0$ for some $a \in \mathfrak{a}$. Conversely, if $(1 - a)x = 0$ for some $x \in M$, then

$$x = ax = a^2 x = \cdots \in \bigcap_{n\geqslant 1} \mathfrak{a}^n M = E$$

$\square$

Corollary 2.4.1. Let A be a Noetherian domain and $\mathfrak{a}$ an proper ideal. Then $\bigcap_{n\geqslant 1} \mathfrak{a}^n = 0$.

Proof. Note that $1 - a \neq 0$. $\square$

Corollary 2.4.2. If $\mathfrak{a}$ is contained in the Jacobson radical of A, then $\bigcap_{n\geqslant 1} \mathfrak{a}^n M = 0$.

Proof. In this case, $1 - a$ is a unit in A, so $\bigcap_{n \geqslant 1} \mathfrak{a}^n M = 0$. $\square$

Corollary 2.4.3. Let A be a Noetherian ring and $\mathfrak{a}$ its the Jacobson radical. Then $\bigcap_{n \geqslant 1} \mathfrak{a}^n = 0$.

Corollary 2.4.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then $\bigcap_{n \geqslant 1} \mathfrak{m}^n = 0$.

Corollary 2.4.5. Let A be a Noetherian ring and $\mathfrak{p} \in \operatorname{Spec}(A)$. Then the intersection of all $\mathfrak{p}$-primary ideals of A is the kernel of $A \to A_{\mathfrak{p}}$.

Proof. Since A is Noetherian, $\sqrt{I}^n \subseteq I$ for some $n \in \mathbb{N}$, so I is $\mathfrak{p}$-primary if and only if $\mathfrak{p}^n \subseteq I \subseteq \mathfrak{p}$ for some $n \in \mathbb{N}$. Note that $I \mapsto IA_{\mathfrak{p}}$ establishes a bijection on $\mathfrak{p}$-primary ideals of A and $\mathfrak{p}_{\mathfrak{p}}$-primary ideals of $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is Noetherian local, $\bigcap_{n \geqslant 1} \mathfrak{p}_{\mathfrak{p}}^n = 0$ in $A_{\mathfrak{p}}$, and hence $\ker(A \to A_{\mathfrak{p}}) = \bigcap_{n \geqslant 1} \mathfrak{p}^n$ $\square$

2.3 Associated Graded Rings

Let A be a ring and $\mathfrak{a}$ an ideal of A. Define

$$\operatorname{Gr}_{\mathfrak{a}}(A) = \bigoplus_{n \geqslant 0} \mathfrak{a}^n / \mathfrak{a}^{n+1}$$

This is a graded ring, in which the multiplication is defined as follows.

- For $x \in \mathfrak{a}^n$, denote by $\overline{x}$ its image in $\mathfrak{a}^n/\mathfrak{a}^{n+1}$. Then for $x \in \mathfrak{a}^n$, $y \in \mathfrak{a}^m$, define $\overline{x}\overline{y} = \overline{xy}$ to be the image of $xy \in \mathfrak{a}^{n+m}$ in $\mathfrak{a}^{n+m}/\mathfrak{a}^{n+m+1}$.
- $\overline{x}\overline{y} = \overline{xy}$ is well-defined since we take modulo $\mathfrak{a}^{n+m+1}$.

Similarly, if M is an A-module and $F := \{M_n\}_{n \geqslant 0}$ is an $\mathfrak{a}$-filtration of M, define

$$\operatorname{Gr}_F(M) := \bigoplus_{n \geqslant 0} M_n / M_{n+1}$$

and denote $\operatorname{Gr}_F^n(M) = M_n/M_{n+1}$. Then $\operatorname{Gr}_F(M)$ is a graded $\operatorname{Gr}_{\mathfrak{a}}(A)$-module.

Proposition 2.5. Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then

1. $\operatorname{Gr}_{\mathfrak{a}}(A)$ is Noetherian;
2. if M is a finite A-module and $F = \{M_n\}_{n \geqslant 0}$ is a stable $\mathfrak{a}$-filtration of M, then $\operatorname{Gr}_F(M)$ is a finite $\operatorname{Gr}_{\mathfrak{a}}(A)$-module.

Proof.

1. Write $\mathfrak{a} = Ax_1 + \cdots + Ax_r$, and denote by $\overline{x_i}$ the image of x_i in $\mathfrak{a}/\mathfrak{a}^2$. Then $\operatorname{Gr}_{\mathfrak{a}}(A) = (A/\mathfrak{a})[\overline{x_1}, \ldots, \overline{x_r}]$ (by its definition). Since $A/\mathfrak{a}$ is Noetherian, $\operatorname{Gr}_{\mathfrak{a}}(A)$ is Noetherian by Hilbert's basis theorem.

2. Say $\mathfrak{a}M_n = M_{n+1}$ for some $n \geqslant 0$. Then

$$\mathrm{Gr}_F(M) = M/M_1 \oplus \cdots \oplus M_{n-1}/M_n \oplus \bigoplus_{k \geqslant 0} \mathfrak{a}^k M_n/\mathfrak{a}^{k+1} M_n$$

is generated by $M/M_1 \oplus \cdots \oplus M_{n-1}/M_n \oplus M_n/M_{n+1}$. Each M_r is a finite A-module, and $\mathrm{Gr}_F^r(M) = M_r/M_{r+1}$ is annihilated by $\mathfrak{a}$, so $\mathrm{Gr}_F^r(M)$ is a finite $A/\mathfrak{a}$-module. Hence $\mathrm{Gr}_F(M)$ is a finite $\mathrm{Gr}_{\mathfrak{a}}(M)$-module.

$\square$

2.3.1 Initial forms

Let A be a ring, $\mathfrak{a}$ an ideal of A, M an A-module and $F := \{M_n\}_{n \geqslant 0}$ an $\mathfrak{a}$-filtration of M.

- For $f \in M$, let $m = \sup\{n \in \mathbb{N}_0 \mid f \in M_n\}$. Define the **initial form of** f by

$$\mathrm{in}(f) := \begin{cases} f \bmod M_{m+1} \in \mathrm{Gr}_F^m(M) & , \text{if } m < \infty \\ 0 & , \text{if } f \in \bigcap_{n \geqslant 1} M_n \end{cases}$$

  Note that $\mathrm{in} : M \to \mathrm{Gr}_F(M)$ is not a homomorphism.

- For a A-submodule M' of M, define

$$\mathrm{in}(M') := \langle \mathrm{in}(f) \mid f \in M' \rangle_{\mathrm{Gr}_{\mathfrak{a}}(A)}$$

  to be the $\mathrm{Gr}_{\mathfrak{a}}(A)$-submodule of $\mathrm{Gr}_F(M)$.

Corollary 2.4.6. Let R be a Noetherian local ring and $\mathfrak{a}$ be an ideal. If $\mathrm{Gr}_{\mathfrak{a}}(R)$ is a domain, then so is R.

Proof. Suppose $fg = 0 \in R$. Then $\mathrm{in}(f)\,\mathrm{in}(g) = 0 \in \mathrm{Gr}_{\mathfrak{a}}(R)$, and thus either $\mathrm{in}(f) = 0$ or $\mathrm{in}(g) = 0$. Since $\bigcap_{n \geqslant 0} \mathfrak{a}^n = 0$, this implies $f = 0$ or $g = 0$. $\square$

3 Dimension

Definition. Let $A \neq 0$ be a ring.

1. A **prime chain** is a strictly increasing sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

   of prime ideals of A. Its length is defined to be n, the number of inclusions.

2. The height $\mathrm{ht}(\mathfrak{p})$ of a prime $\mathfrak{p}$ is the supremum of the lengths of all prime chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

   More generally, for an proper ideal I of A, define

$$\mathrm{ht}(I) := \inf\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in V(I)\} = \inf\{\mathrm{ht}(\mathfrak{p}) \mid I \subseteq \mathfrak{p}\}$$

3. The **Krull dimension** of A is defined as

$$\dim A := \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}$$

- By definition, $\mathrm{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$ for all primes $\mathfrak{p}$.
- For any ideals I,

$$\dim(A/I) + \mathrm{ht}(I) \leqslant \dim A$$

For $M \neq 0$ an A-module, define its **dimension** to be

$$\dim M = \dim(A/\operatorname{ann}_A(M))$$

3.1 Length

Proposition 3.1. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then

(i) M is Noetherian if and only if M', M'' are Noetherian.

(ii) M is Artinian if and only if M', M'' are Artinian.

Proof. A chain of M', M'' gives rise to a chain in M; this shows the only if part of both statements. Conversely, a chain in M restricts to a chain in M', and maps to a chain in M''; this shows the if part. □

Corollary 3.1.1. Finite direct sums of Noetherian (resp. Artinian) A-modules are Noetherian (resp. Artinian).

Corollary 3.1.2. Let A be a Notherian (resp. Artinian) ring. If M is a finite A-module, then M is Notherian (resp. Artinian).

Corollary 3.1.3. Quotients of Noetherian (resp. Artinian) rings are Noetherian (resp. Artinian).

Definition. Let A be a ring and M an A-module.

1. The **length of the chain** of submodules of M
$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n$$
is defined to be n.

2. A **finite chain / finite filtration** of submodules of A-module M is the chain of the form
$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

3. A **composition series** of M is a maximal finite chain of M, i.e., each successive quotient module is a simple module.

4. The **length** of M is defined the be the minimal length of composition series of M, and is denoted by $\text{length}_A(M)$; if M does not possess a composition series, define $\text{length}_A(M) = +\infty$.

If M has a composition, we also use the term M **has finite length**.

- It's clear that every chain can be extended to a finite chain by adding 0 in the end of the chain, if $M_n \neq 0$.

Proposition 3.2. Suppose M has a composition series. Then every composition series has the same length. Moreover, every chain can be extended to a composition series of M.

Proof.

1° $N \subsetneq M \Rightarrow \text{length}_A(N) < \text{length}_A(M)$. Let $M = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = 0$ be a composition series of M. Consider the chain
$$N = M_0 \cap N \supseteq M_1 \cap N \supseteq M_{n-1} \cap N \supseteq M_n \cap N = 0$$

- Note that $\dfrac{M_i \cap N}{M_{i+1} \cap N} \subseteq \dfrac{M_i}{M_{i+1}}$; since the latter is simple, $\dfrac{M_i \cap N}{M_{i+1} \cap N} = 0$ or $\dfrac{M_i \cap N}{M_{i+1} \cap N}$ is simple.

Hence $\text{length}_A(N) \leqslant \text{length}_A(M)$, and equality holds if and only if every $\dfrac{M_i \cap N}{M_{i+1} \cap N}$ is simple. Starting from $M_{n-1} = M_{n-1} \cap N \Leftrightarrow M_{n-1} \subseteq N$, we can show that $M = M_0 \subseteq N \subseteq M$, i.e., $N = M$.

2° Every chain in M has length $\leqslant \text{length}_A(N)$. Take a chain of length n in M

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n$$

Then

$$\text{length}_A(M) > \text{length}_A(M_1) > \cdots > \text{length}_A(M_n) \geqslant 0$$

and hence $\text{length}_A(M) \geqslant n$.

3° Every composition has length $\text{length}_A(M)$. This follows from definition and 2°.

4° A chain in M of length $\text{length}_A(M)$ is a composition series. Firstly, such a chain must be a finite chain. Secondly, if any successive quotient is not simple, we can insert a submodule to lengthen the chain; taking the contrapositive, since such a chain can not be lengthened by 2°, it must be a composition series.

5° Every chain in M can be extended to a composition series in M. This follows from 2° and 4°.

□

Corollary 3.2.1. *M has finite length if and only if M is Artinian and Noetherian.*

Proof. The only if part is clear. Suppose M is Artinian and Noetherian. Since M is Noetherian, we can construct a descending chain of submodules of M

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n$$

with the property that M_i is a maximal submodule contained in M_{i-1}. Since M is Artinian, this chain must stop, i.e., M has a composition series. □

Corollary 3.2.2. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Then M has finite length if and only if M', M'' have finite length. Moreover, we have

$$\text{length}_A(M) = \text{length}_A(M') + \text{length}_A(M'')$$

Proof. The first assertion follows from the previous Corollary and Proposition 3.1. It remains to show that equality. But we can use a composition series in M' and that in M'' to produce a composition series in M, which shows that equality. □

Corollary 3.2.3. For a k-vector space V, TFAE:

- $\dim_k V < \infty$
- $\text{length}_k(V) < \infty$.
- V is Artinian.
- V is Noetherian.

Proof. It remains to show that if V is infinite dimensional, then V is neither Artinian nor Noetherian. Let $\{v_1, v_2, \ldots, \}$ be a k-basis for V. Form the subspaces

$$W_n = \text{span}\{v_1, \ldots, v_n\}, \qquad U_n = \text{span}\{v_n, v_{n+1}, \ldots\}$$

Then $\{W_n\}$ is ascending and $\{U_n\}$ is descending, each of which is not stable. □

We will use the following corollary to characterize all Artinian rings. First note that if M is an A-module and I is an ideal contained in $\mathrm{Ann}_A(M)$. Then M has a natural A/I-module structure, and

$$M \text{ is Noetherian (resp. Artinian) as an } A\text{-module} \Leftrightarrow M \text{ is Noetherian (resp. Artinian) as an } A/I\text{-module.}$$

This is because for an subgroup N of M, being an A/I-submodule of M and being an A-submodule of M are equivalent for N. Also, $\mathrm{length}_A(M) = \mathrm{length}_{A/I}(M)$.

Corollary 3.2.4. Let A be a ring in which the zero ideal is a product $\mathfrak{m}_1 \cdots \mathfrak{m}_r$ of (not necessarily distinct) maximal ideals. Then A is Noetherian if and only if A is Artinian.

Proof. Consider the chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_r = 0$$

Every successive quotient has a natural structure of A/m_i-vector spaces. Hence each successive quotient is Noetherian if and only if it is Artinian. Repeated uses of Proposition 3.1 then shows that A is Noetherian if and only if A is Artinian. ☐

3.1.1 Characterization of Artinian Rings

Proposition 3.3. Let A be a nontrivial Artinian ring.

1. A is a **semilocal ring**, i.e., $\#\mathrm{mSpec}(A) < \infty$.

2. The zero ideal of A is a product of maximal ideals.

3. A is a finite product of Artinian local rings.

4. A is Noetherian.

5. Every prime is maximal.

Proof.

1. Consider the collection of finite intersections of maximal ideals of A; since A is Artinian, it contains a minimal element, say $I := \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. Then for every $\mathfrak{m} \in \mathrm{mSpec}(A)$, $\mathfrak{m} \cap I = I$ by minimality, and thus $I \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ is a prime, $\mathfrak{m}_i = \mathfrak{m}$ for some i. Hence $\mathrm{mSpec}(A) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$.

2. Let J be the product of all maximal ideals of A. Consider the descending chain

$$J \supseteq J^2 \supseteq \cdots \supseteq J^\ell \supseteq$$

Since A is Artinian, there exists $m \in \mathbb{N}_0$ with $J^m = J^{m+1}$. We show $J^m = 0$. Suppose $J^m \neq 0$; consider the collection $\{0 \neq I \trianglelefteq R \mid IJ^m \neq 0\}$. This collection is nonempty for $JJ^m = J^{m+1} = J^m \neq 0$, and hence it has a minimal element, say I_0. Let $f \in I_0$ with $fJ^m \neq 0$; since I is minimal, $I = fA$. Also, $(fJ)J^m = fJ^m \neq 0$, so $fJ = fA$ by minimality. Now $fr = f$ for some $r \in J$, i.e., $(1-r)f = 0$. Since J is the Jacobson radical, $1 - r \in A^\times$, implying $f = 0$, a contradiction. Therefore $J^m = 0$.

3. Let $J = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$, where the $\mathfrak{m}_i$ are all the maximal ideals of A. Then $J^m = 0$ for some m by 2. By Chinese Remainder theorem, we have $A = A/J^m \cong \prod_{i=1}^{r} A/\mathfrak{m}_i^m$. Since A is Artinian, each $A/\mathfrak{m}_i^m$ is Artinian. Also, $A/\mathfrak{m}_i^m$ is local with the maximal ideal $\mathfrak{m}_i/\mathfrak{m}_i^m$.

4. This follows from 2. and Corollary 3.2.4.

5. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. We must show for every $f \notin \mathfrak{p}$, $(f, \mathfrak{p}) = A$. Consider the chain

$$(f, \mathfrak{p}) \supseteq (f^2, \mathfrak{p}) \supseteq \cdots \supseteq (f^n, \mathfrak{p}) \supseteq \cdots$$

Then $(f^m, \mathfrak{p}) = (f^{m+1}, \mathfrak{p})$ for some $m \in \mathbb{N}$ so that $f^m = rf^{m+1} + a$ for some $r \in A$, $a \in \imath$. Hence $f^m(1 - rf) \in \mathfrak{p}$; since $f^m \neq \mathfrak{p}$, $1 - rf \in \mathfrak{p}$, and thus $1 \in (f, \mathfrak{p})$, as wanted.

□

Theorem 3.4. Let A be a ring. TFAE:

1. A has a composition series as an A-module.

2. A is Artinian.

3. A is Noetherian and every prime ideal is maximal.

Proof. We prove the equivalence in the order 1. ⇒ 2. ⇒ 3. ⇒ 1. We already see 1. ⇒ 2. ⇒ 3. It remains to show 3. ⇒ 1. Suppose otherwise that A does not have finite length; consider the collection $\{I \trianglelefteq A \mid \operatorname{length}_A(A/I) = \infty\}$. This is nonempty since the zero ideal belongs to it, and hence it has a maximal element, say $\mathfrak{p}$. We claim $\mathfrak{p}$ is a prime. Let $a, b \in A$ with $ab \in \mathfrak{p}$, $a \notin \mathfrak{p}$; then we may form the exact sequence

$$0 \longrightarrow A/(\mathfrak{p} : a) \xrightarrow{\ a\ } A/\mathfrak{p} \longrightarrow A/(a, \mathfrak{p}) \longrightarrow 0$$

If $b \notin \mathfrak{p}$, then $\mathfrak{p} \subsetneq (\mathfrak{p} : a)$; since both $(\mathfrak{p} : a)$ and $(a, \mathfrak{p})$ properly contain $\mathfrak{p}$, both $A/(\mathfrak{p} : a)$ and $A/(a, \mathfrak{p})$ have finite length by maximality, and so does $A/\mathfrak{p}$ by Corollary 3.2.2, a contradiction. Hence $b \in \mathfrak{p}$, and so $\mathfrak{p}$ is a prime; moreover, $\mathfrak{p}$ is maximal by assumption. However, it turns out $A/\mathfrak{p}$ is a field, which has finite length, contradicting the definition of $\mathfrak{p}$. Hence A has finite length. □

Corollary 3.4.1. Let A be a Noetherian and $M \neq 0$ a finite A-module. TFAE:

1. M has finite length as an A-module.

2. The ring $A/\operatorname{ann}_A(M)$ is Artinian.

3. $\dim M = 0$.

3.2 Hilbert's Polynomials

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded Noetherian ring; then A_0 is Noetherian and $A = A_0[x_1, \ldots, x_s]$ with each $x_i \in A_+$ homogeneous of degree $d_i > 0$. Let M be a finitely generated graded A-module. Then M can be generated by a finite number of homogeneous elements, say $m_1, \ldots, m_t$ with degree $r_1, \ldots, r_t$, respectively.

- Each M_n is a finite A_0-module. Indeed, each element in M_n can be written as a sum $\sum_{i=1}^{t} f_j(x)m_j$ with $f_j(x) \in A$ homogeneous of degree $n - r_j$, so M_n is generated by all $g_j(x)m_j$, where $g_j(x)$ is a monomial in the x_j of total degree $n - r_j$.

Let λ be an additive function (with valued in $\mathbb{Z}$) on all finite A_0-modules, namely, for all short exact sequences $0 \to M' \to M \to M'' \to 0$ of finite A_0-modules, we have

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0$$

The **Poincaré series** $P(M,t) = P_\lambda(M,t)$ of M is the generating function of $\lambda(M_n)$

$$P(M,t) := \sum_{n=0}^{\infty} \lambda(M_n)t^n$$

Example. Let $A = A_0[x_1, \ldots, x_s]$, where A_0 is an Artinian ring and the x_i are independent variables. Then A_n is a free A_0-module with generated by the monomial $x_1^{m_1} \cdots x_s^{m_s}$ with $m_1 + \cdots + m_s = n$; they are in number $\binom{s+n-1}{s-1}$, and hence

$$P(A,t) := \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} t^n = (1-t)^{-s}$$

Theorem 3.5 (Hilbert, Serre)**.** *$P(M,t)$ is a rational function in t of the form $\dfrac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$, where $f(t) \in \mathbb{Z}[t]$.*

Proof. Use induction on s, the number of the generators of A over A_0.

- $s = 0$. Then $M = A_0m_1 + \cdots + A_0m_t$, so $M_n = 0$ for $n > \max\{r_1, \ldots, r_t\}$. Hence $P(M,t)$ is just a polynomial in t.

- $s > 0$. Multiplying M_n by x_s gives an exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{x_s} M_{n+k_s} \longrightarrow L_{n+k_s} \longrightarrow 0 \tag{1}$$

  Define $K = \bigoplus_n K_n$, $L = \bigoplus_n L_n$; being a submodule and a quotient of M, K and L are finite A-module. Since they are annihilated by x_s, they are graded $A_0[x_1, \ldots, x_{s-1}]$-modules.

Applying λ to (1), we get

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

Multiplying by t^{n+k_s} and summing over n, we obtain (for some $g \in \mathbb{Z}[t]$)

$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + P(M,t) - P(L,t) = g(t)$$

or $(1-t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t)$. The result then follows by induction.

□

- Let $d(M) = \operatorname{ord}_{t=1} P(M,t)$ be the order of $P(M,t)$ at $t = 1$.

Corollary 3.5.1. If $x \in A_k$ is a non-zero-divisor in M, then $d(M/xM) = d(M) - 1$.

Proof. Replacing x_s in (1) with x, we see $K = 0$ and $L = M/xM$. Hence $(1-t^k)P(M,t) = P(M/xM,t) + g(t)$ for some $g \in \mathbb{Z}[t]$, so $d(M) - 1 = d(M/xM)$. □

Corollary 3.5.2. If each $k_i = 1$. then for $n \gg 0$, $\lambda(M_n)$ is a polynomial in n with rational coefficients of degree[1] $d-1$ ($d = d(M)$), called the **Hilbert's polynomial of** M.

Proof. By Theorem, $\sum_{n=0}^{\infty} \lambda(M_n)t^n = f(t)(1-t)^{-s}$; we may assume $s = d$ and $f(1) \neq 0$. Write $f(t) = \sum_{k=0}^{N} a_k t^k$ with $a_N \neq 0$. Then for $n \geqslant N$,

$$\lambda(M_n) = \sum_{k=0}^{N} a_k \binom{d+n-k-1}{d-1}$$

with the convention $\binom{n}{-1} = 0$ for $n \geqslant 0$ and $\binom{-1}{-1} = 1$. The leading term is

$$a_N \frac{n^{d-1}}{(d-1)!} \neq 0$$

so $\lambda(M_n)\,(n \geqslant N)$ has degree $d-1$. □

Proposition 3.6. Let $(A, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, M a finitely generated A-module and $F = \{M_n\}_{n \geqslant 0}$ a stable $\mathfrak{q}$-filtration on M. Then

1. M/M_n has finite length for all $n \geqslant 0$;
2. for $n \gg 0$ this length is a polynomial $g(n)$ in n of degree $\leqslant s$, where s is the least number of generators of $\mathfrak{q}$;
3. the degree and leading coefficient of $g(n)$ depend only on M and $\mathfrak{q}$, not on the chosen filtration.

[1]For this statement, the zero polynomial is assumed to have degree -1.

Proof.

1. Let $\mathrm{Gr}_{\mathfrak{q}}(A) = \bigoplus_{n\geqslant 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ and $\mathrm{Gr}_F(M) = \bigoplus_{n\geqslant 0} M_n/M_{n+1}$. Note that $\mathrm{Gr}^0_{\mathfrak{q}}(A) = A/\mathfrak{q}$ is Artinian local, $\mathrm{Gr}_{\mathfrak{q}}(A)$ is Noetherian and $\mathrm{Gr}_F(M)$ is a finite $\mathrm{Gr}_{\mathfrak{q}}(A)$-module. Each $\mathrm{Gr}^n_F(M) = M_n/M_{n+1}$ is a Noetherian finite A-module annihilated by $\mathfrak{q}$, so hence a finite $A/\mathfrak{q}$-module. Since $A/\mathfrak{q}$ is Artinian, $\mathrm{Gr}^n_F(M)$ is Artinian; hence $\mathrm{Gr}^n_F(M)$ has finite length, and so does M/M_n with
$$l_n := \mathrm{length}_A(M/M_n) = \sum_{r=1}^{n} \mathrm{length}_A(M_{r-1}/M_r)$$

2. Say $\mathfrak{q} = Ax_1 + \cdots + Ax_s$; then $\mathrm{Gr}_{\mathfrak{q}}(A) = (A/q)[\overline{x_1}, \ldots, \overline{x_s}]$, where $\overline{x_i}$ denotes the image of x_i in $\mathfrak{q}/\mathfrak{q}^2$. By Corollary 3.5.2, $\mathrm{length}_A(M_n/M_{n+1}) = f(n)$, where f is a polynomial in n of degree $\leqslant s-1$ for all large n. We have $l_{n+1} - l_n = f(n)$, so l_n itself is a polynomial $g(n)$ of degree $\leqslant s$ for all large n.

3. Let $\{\tilde{M}_n\}_{n\geqslant 0}$ be another stable $\mathfrak{q}$-filtration of M, and let $\tilde{g}(n) = \mathrm{length}_A(M/\tilde{M}_n)$. $\{\tilde{M}_n\}$ and $\{M_n\}$ being stable, there exists $n_0 \in \mathbb{N}$ such that $M_{n+n_0} \subseteq \tilde{M}_n$ and $\tilde{M}_{n+n_0} \subseteq M_n$ for all $n \geqslant 0$; hence
$$g(n+n_0) \geqslant \tilde{g}(n), \qquad \tilde{g}(n+n_0) \geqslant g(n)$$
For $n \gg 0$, both g and $\tilde{g}$ are polynomials in n, so $\lim_{n\to\infty} \dfrac{g(n)}{\tilde{g}(n)} = 1$, meaning that they have the same degree and the leading coefficient.

$\square$

- The polynomial $g(n)$ corresponding to the filtration $\{\mathfrak{q}^n M\}_{n\geqslant 0}$ is denoted by
$$\chi^M_{\mathfrak{q}}(n) := \mathrm{length}_A(M/\mathfrak{q}^n M) \quad (n \gg 0)$$
- If $M = A$, we write $\chi_{\mathfrak{q}}(n)$ for $\chi^M_{\mathfrak{q}}(n)$, and call it the **characteristic polynomial** of the $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Corollary 3.6.1. For $n \gg 0$, the length $\mathrm{length}_A(A/\mathfrak{q}^n)$ is a polynomial $\chi_{\mathfrak{q}}(n)$ of degree $\leqslant s$, where s is the least number of generators of $\mathfrak{q}$.

Corollary 3.6.2. If $A, \mathfrak{m}, \mathfrak{q}$ are as above, the $\deg \chi_{\mathfrak{q}}(n) = \deg \chi_{\mathfrak{m}}(n)$.

Proof. Since A is Noetherian, $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some $r \in \mathbb{N}$, and hence
$$\mathfrak{m}^{rn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n$$
for all $n \in \mathbb{N}$, implying for all large n
$$\chi_{\mathfrak{m}}(n) \leqslant \chi_{\mathfrak{q}}(n) \leqslant \chi_{\mathfrak{m}}(rn)$$
Taking $n \to \infty$, we see $1 \leqslant \lim_{n\to\infty} \dfrac{\chi_{\mathfrak{q}}(n)}{\chi_{\mathfrak{m}}(n)} < \infty$, so they have the same degree. $\square$

- We denote by $d(A)$ the common degree of the $\chi_{\mathfrak{q}}(n)$. Then

$$d(A) = d(\mathrm{Gr}_{\mathfrak{m}}(A))$$

where $d(\mathrm{Gr}_{\mathfrak{m}}(A))$ is defined before to be the order of the pole at $t = 1$ of the Hilbert polynomial of $\mathrm{Gr}_{\mathfrak{m}}(A)$.

3.3 Noetherian Local Rings

Let $(A, \mathfrak{m})$ be a Noetherian local rings. Define

$$\delta(A) = \text{least number of the generators of an } \mathfrak{m}\text{-primary ideal of } A$$

We will prove $\delta(A) = d(A) = \dim(A)$, by proving

$$\delta(A) \geqslant d(A) \geqslant \dim(A) \geqslant \delta(A)$$

By the last two Corollary, we obtain

Proposition 3.7. $\delta(A) \geqslant d(A)$.

Proposition 3.8. Let $A, \mathfrak{m}, \mathfrak{q}$ as before, M is a finite A-module, $x \in A$ a non-zero-divisor in M and $M' = M/xM$. Then

$$\deg \chi_{\mathfrak{q}}^{M'} \leqslant \deg \chi_{\mathfrak{q}}^{M} - 1$$

Proof. Let $N = xM$; then $N \cong M$ as A-modules, for $x \in A$ is a non-zero-divisor in M. Define $N_n = N \cap \mathfrak{q}^n M$. Then we have an exact sequence

$$0 \longrightarrow N/N_n \longrightarrow M/\mathfrak{q}^n M \longrightarrow M'/\mathfrak{q}^n M' \longrightarrow 0$$

Hence, if $g(n) = \mathrm{length}_A(N/N_n)$, then

$$g(n) - \chi_{\mathfrak{q}}^{M}(n) + \chi_{\mathfrak{q}}^{M'}(n) = 0$$

for $n \gg 0$. By Artin-Ree's lemma, $\{N_n\}_{n \geqslant 0}$ is a stable $\mathfrak{q}$-filtration of N. By Proposition 3.6 3, g and $\chi_{\mathfrak{q}}^{M}$ have the same degree and the leading coefficient, hence proved. $\square$

The following corollary is an analog of Corollary 3.5.1 for Noetherian local rings.

Corollary 3.8.1. If A is a Noetherian local ring, x a regular element in A, then $d(A/(x)) \leqslant d(A) - 1$.

Proposition 3.9. $d(A) \geqslant \dim A$.

Proof. Induction on $d(A)$. When $d(A) = 0$, then $\text{length}_A(A/\mathfrak{m}^n)$ is constant for $n \gg 0$, and hence $\mathfrak{m}^n = \mathfrak{m}^{n+1}$; by Nakayama's lemma, $\mathfrak{m}^n = 0$, so $A \cong A/\mathfrak{m}^n$ is Artinian and thus $\dim A = 0$.

Assume $d(A) > 0$, and let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be any prime chain in A. Let $x \in \mathfrak{p}_1 \backslash \mathfrak{p}_0$; put $A' = A/\mathfrak{p}_0$ and x' to be the image of x in A'. Then A' is an integral domain and $x' \neq 0$. By above Corollary,

$$d(A'/(x')) \leqslant d(A') - 1$$

On the other hand, if $\mathfrak{m}'$ is the maximal ideal of A', then $A'/\mathfrak{m}'^n$ is a homomorphic image of $A/\mathfrak{m}^n$, so

$$\text{length}_{A'}(A'/\mathfrak{m}'^n) = \text{length}_A(A'/\mathfrak{m}'^n) \leqslant \text{length}_A(A/\mathfrak{m}^n)$$

implying $d(A') \leqslant d(A)$. Therefore $d(A'/(x')) \leqslant d(A) - 1$, and by the induction hypothesis, we have

$$\dim A'/(x') \leqslant d(A'/(x')) \leqslant d(A) - 1$$

But images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in $A'/(x')$ form a prime chain of length $r - 1$, so $r - 1 \leqslant d(A) - 1$, or $r \leqslant d(A)$; hence $\dim A \leqslant d(A)$. $\square$

Corollary 3.9.1. If A is a Notherian local ring, $\dim A$ is finite.

Corollary 3.9.2. In a Noetherian ring every prime has finite height. In particular, the set of prime ideals in a Noetherian ring satisfies the descending chain condition.

Proposition 3.10. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then there exists an $\mathfrak{m}$-primary ideal in A generated by d elements $x_1, \ldots, x_d$, and therefore $\dim A \geqslant \delta(A)$.

Proof. We construct $x_1, \ldots, x_d$ inductively so that every prime ideal containing $(x_1, \ldots, x_i)$ has height $\geqslant i$ for each i. Suppose $i > 0$ and $x_1, \ldots, x_{i-1}$ have been constructed. Let $\mathfrak{p}_j$ $(1 \leqslant j \leqslant s)$ be the minimal prime ideals (if any) of $(x_1, \ldots, x_{i-1})$ which have height exactly $i - 1$. Since

$$i - 1 < d = \dim A = \dim A_{\mathfrak{m}} = \text{ht}(\mathfrak{m})$$

we have $\mathfrak{m} \neq \mathfrak{p}_j$ $(1 \leqslant j \leqslant s)$, hence $\mathfrak{m} \neq \bigcup_{j=1}^{s} \mathfrak{p}_j$; choose $x_i \in \mathfrak{m} \backslash \bigcup_{j=1}^{s} \mathfrak{p}_j$. Let $\mathfrak{q}$ be a prime ideal containing $(x_1, \ldots, x_i)$. Then $\mathfrak{q} \supseteq \mathfrak{p}_j$ for some $1 \leqslant j \leqslant s$; since $x_i \in \mathfrak{q} \backslash \mathfrak{p}_j$, we have $\text{ht}(\mathfrak{q}) > \text{ht}(\mathfrak{p}_j) = i - 1$, or $\text{ht}(\mathfrak{q}) \geqslant i$, as shown. $\square$

Theorem 3.11. For any Noetherian local ring $(A, \mathfrak{m})$ the following three integers are equal:

(i) the maximum length of prime chains in A.

(ii) the degree of the characteristic polynomial $\chi_{\mathfrak{m}}(n) = \text{length}_A(A/\mathfrak{m}^n)$.

(iii) the least number of generators of an $\mathfrak{m}$-primary ideal of A.

Example. Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $\mathfrak{m} = (x_1, \ldots, x_n)$. Then $\mathrm{Gr}_{\mathfrak{m}}(A_{\mathfrak{m}}) \cong k[x_1, \ldots, x_n]$, so its Poincaré series is $(1-t)^{-n}$ by Example in the beginning of the section. Hence $\dim A_{\mathfrak{m}} = n$.

Corollary 3.11.1. $\dim A \leqslant \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, where $k = A/\mathfrak{m}$ is the residue field.

Proof. Let $x_1, \ldots, x_s$ be elements of $\mathfrak{m}$ such that their images in $\mathfrak{m}/\mathfrak{m}^2$ form an k-basis. By Nakayama's lemma, $x_1, \ldots, x_s$ generates $\mathfrak{m}$. Hence $\dim A \leqslant s = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. □

Corollary 3.11.2. Let A be a Noetherian ring and $x_1, \ldots, x_r \in A$. Then every minimal prime $\mathfrak{p}$ belonging to $(x_1, \ldots, x_r)$ has height $\leqslant r$.

Proof. In $A_{\mathfrak{p}}$, the ideal $(x_1, \ldots, x_r)$ becomes $\mathfrak{p}$-primary, so $r \geqslant \dim A_{\mathfrak{p}} = \mathrm{ht}(\mathfrak{p})$. □

Corollary 3.11.3 (Krull's PIT)**.** Let A be a Noetherian ring and let $x \in A$ be neither a unit nor a zero-divisor. Then every minimal prime $\mathfrak{p}$ of (x) has height 1.

Proof. By above Corollary, $\mathrm{ht}(\mathfrak{p}) \leqslant 1$. If $\mathrm{ht}(\mathfrak{p}) = 0$, then $A_{\mathfrak{p}}$ is Artinian local, so $\mathfrak{p}_{\mathfrak{p}}^n = 0$ for some $n \in \mathbb{N}$. Thus $x^n/1 = 0$ in $A_{\mathfrak{p}}$, and thus $ax^n = 0$ for some $a \in A - \mathfrak{p}$; but this means x is a zero-divisor, a contradiction. □

Corollary 3.11.4. Let $(A, \mathfrak{m})$ be a Noetherian local ring and $x \in \mathfrak{m}$ which is not a zero-divisor. Then $\dim A/(x) = \dim A - 1$.

Proof. Put $d = \dim A/(x)$. By Corollary 3.8.1, $d \leqslant \dim A - 1$. Conversely, let $x_1, \ldots, x_d \in A$ that generates an $\mathfrak{m}/(x)$-primary ideal in $A/(x)$. Then $(x, x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary in A, hence $d + 1 \geqslant \dim A$. □

Corollary 3.11.5. Let A be a Notherian ring and $\mathfrak{p}$ be a prime ideal of height r. Then $\mathfrak{p}$ is minimal over an ideal generated by r elements.

Proof. Passing to $(A_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}})$, with Proposition 3.10, we can find a $\mathfrak{p}_{\mathfrak{p}}$-primary ideal $(x_1, \ldots, x_r)$ of $A_{\mathfrak{p}}$, where $x_1, \ldots, x_r \in A_{\mathfrak{p}}$, with the property that every prime containing $(x_1, \ldots, x_r)$ has height $\geqslant r$; up to units we may assume $x_1, \ldots, x_r \in A$. Then $\mathfrak{p} \supseteq (x_1, \ldots, x_r) \cap A \supseteq \mathfrak{p}^n$ for some n. Then $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_r) \cap A$. □

Corollary 3.11.6. A Noetherian domain A is a UFD if every height one prime ideal of A is principal.

Proof. By above corollaries, height one primes are precisely those primes minimal over principal ideals. Then apply Proposition 1.17. □

3.4 Systems of Parameters

Definition. If $(x_1, \dots, x_d) \in A$ generates an $\mathfrak{m}$-primary ideal with $d = \dim A$, we call $x_1, \dots, x_d$ a **system of parameters**.

Proposition 3.12. Let $x_1, \dots, x_d$ be a system of parameter for $(A, \mathfrak{m})$ and let $\mathfrak{q} = (x_1, \dots, x_d)$ be the $\mathfrak{m}$-primary ideal generated by them. Let $f(t_1, \dots, t_d)$ be a homogeneous polynomial of degree s with coefficients in A, and assume that

$$f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}$$

Then all the coefficients of f lie in $\mathfrak{m}$.

Proof. Consider the surjective homomorphism

$$\begin{array}{ccc} (A/\mathfrak{q})[t_1, \dots, t_d] & \xrightarrow{\quad\alpha\quad} & \mathrm{Gr}_{\mathfrak{q}}(A) \\ t_i & \longmapsto & \overline{x_i} = x_i \bmod \mathfrak{q} \in \mathrm{Gr}^1_{\mathfrak{q}}(A) \end{array}$$

The hypothesis means $f(t_1, \dots, t_d) \bmod \mathfrak{q} \in \ker \alpha$.

- All coefficients lie in $\mathfrak{m}$. Great.
- Some coefficient of f is a unit in A. Then $f \bmod \mathfrak{q}$ is not a zero-divisor, so

$$\begin{aligned} d = d(\mathrm{Gr}_{\mathfrak{q}} A) &\leqslant d((A/\mathfrak{q})[t_1, \dots, t_d]/(f \bmod \mathfrak{q})) \\ &= d((A/\mathfrak{q})[t_1, \dots, t_d]) - 1 \\ &= d - 1 \end{aligned}$$

  by Corollary 3.5.1 and the Example in the beginning, which is a contradiction.

$\square$

Corollary 3.12.1. If $k \subseteq A$ is a field mapping isomorphically onto $A/\mathfrak{m}$, and $x_1, \dots, x_d$ is a system of parameters, then $x_1, \dots, x_d$ are algebraically independent over k.

Proof. Assume $f(x_1, \dots, x_d) = 0$, where f is a polynomial over k. If $f \not\equiv 0$, write $f = f_s +$ h.o.t., with $f_s \not\equiv 0$ homogeneous of degree s. Then $f_s(x_1, \dots, x_d) = 0 \in \mathfrak{q}^{s+1}$; the Proposition above implies all coefficients of f_s lie in $\mathfrak{m}$, which implies $f_s \equiv 0$, a contradiction. $\square$

3.5 Regular Local Rings

Theorem 3.13. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d with $k = A/\mathfrak{m}$. TFAE:

1. $\mathrm{Gr}_{\mathfrak{m}}(A) \cong k[t_1, \dots, t_d]$ as graded k-modules, where the t_i are independent variables.
2. $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$.

3. $\mathfrak{m}$ can be generated by d elements.

If one of the above statements holds, we say A is a **regular local ring**.

Proof. 1. $\Rightarrow$ 2. is clear since $\mathfrak{m}/\mathfrak{m}^2 = kt_1 + \cdots + kt_d$. 2. $\Rightarrow$ 3. follows from Nakayama's lemma. 3. $\Rightarrow$ 1. follows from Proposition 3.12:

- $\alpha : k[x_1, \ldots, x_d] \to \mathrm{Gr}_{\mathfrak{m}}(A)$ is an isomorphism as graded k-modules. It suffices to show

$$f(x_1, \ldots, x_d) \bmod \mathfrak{m} = 0 \Rightarrow f(x_1, \ldots, x_d) \in \mathfrak{m}[x_1, \ldots, x_d]$$

  for all homogeneous $f(x_1, \ldots, x_d)$ over A of degree $s \geqslant 0$. But

$$0 = f(x_1, \ldots, x_d) \bmod \mathfrak{m} \in \mathrm{Gr}^s_{\mathfrak{m}}(A)$$

  implies $f(x_1, \ldots, x_d) \in \mathfrak{m}^{s+1}$, and thus f has coefficients in $\mathfrak{m}$, i.e., $f(x_1, \ldots, x_d) \in \mathfrak{m}[x_1, \ldots, x_d]$.

□

Corollary 3.13.1. A regular local ring is an integral domain.

Proof. Follow from the Theorem and Corollary 2.4.6. □

Corollary 3.13.2. Regular local rings of dimension 1 are precisely discrete valuation rings.

Proof. Recall for a Noetherian local domain $(A, \mathfrak{m}, k)$ of dimension 1, TFAE:

- A is a DVR.
- A is integrally closed.
- $\mathfrak{m}$ is principal.
- $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
- Every nonzero ideal is a power of $\mathfrak{m}$.
- There exists $x \in A$ such that every nonzero ideal is of the form (x^k), $k \geqslant 0$.

□

Example. Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $\mathfrak{m} = (x_1, \ldots, x_m)$. Then $A_{\mathfrak{m}}$ is a regular local ring, for $\mathrm{Gr}_{\mathfrak{m}}(A_{\mathfrak{m}}) \cong k[x_1, \ldots, x_n]$.

3.6 Homomorphisms and Dimension

Definition. Let $\varphi : A \to B$ be a ring homomorphism. For each $\mathfrak{p} \in \operatorname{Spec} A$, the set

$$\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})) = \operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$$

is called the **fibre** over $\mathfrak{p}$.

- There exists a canonical homeomorphism

  $$\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})) \cong \operatorname{Spec}(\phi)^{-1}(\mathfrak{p})$$

  where $\operatorname{Spec}(\phi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is the canonical map.

- If $\mathfrak{P} \in \operatorname{Spec}(B)$ is a prime lying over $\mathfrak{p}$, denote by $\mathfrak{P}^*$ the prime $\mathfrak{P}B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \in \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$. Then

  $$B_{\mathfrak{P}} = (B_{\mathfrak{p}})_{\mathfrak{P}B_{\mathfrak{p}}}$$

  so

  $$(B \otimes_A \kappa(\mathfrak{p}))_{\mathfrak{P}^*} = B_{\mathfrak{P}} \otimes_A \kappa(\mathfrak{p}) = B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$$

Theorem 3.14. Let $\varphi : A \to B$ be a ring homomorphism between Noetherian rings. Let $\mathfrak{P} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{P} \cap A$.

1. $\operatorname{ht}(\mathfrak{P}) \leqslant \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{P}/\mathfrak{p}B)$; in other words, $\dim(B_{\mathfrak{P}}) \leqslant \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{P}} \otimes_A \kappa(\mathfrak{p}))$

2. The equality holds if the going-down theorem holds for φ.

3. If $f = \operatorname{Spec}(\phi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective, and if the going-down theorem holds, then

   (a) $\dim(B) \geqslant \dim(A)$

   (b) $\operatorname{ht}(I) = \operatorname{ht}(IB)$ for any ideal I of A.

Proof.

1. Replacing A and B by $A_{\mathfrak{p}}$ and $B_{\mathfrak{P}}$, we may assume $(A, \mathfrak{p})$ and $(B, \mathfrak{P})$ are local rings with $\mathfrak{P} \cap A = \mathfrak{p}$. Then we must show $\dim(B) \leqslant \dim(A) + \dim(B/\mathfrak{p}B)$. Let $a_1, \ldots, a_r$ be a system of parameters of A and I be the $\mathfrak{p}$-primary ideal of A generated by them. Then $\dim(B/\mathfrak{p}B) = \dim(B/IB)$. Indeed if $\mathfrak{q}$ is a prime of B containing IB, the $\mathfrak{q} \cap A$ contains I. Taking radical we see $\mathfrak{q} \cap A$ contains $\mathfrak{p}$, and thus $\mathfrak{q} \supseteq \mathfrak{p}$. If $\dim(B/IB) = s$ and $\{\overline{b_1}, \ldots, \overline{b_s}\}$ be a system of parameters of B/IB, then $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$ generates a $\mathfrak{P}$-primary ideal of B. Hence $\dim(B) \leqslant r + s$.

2. Use the notation as above. If $\operatorname{ht}(\mathfrak{P}/\mathfrak{p}B) = s$, there exists a prime chain

   $$\mathfrak{P} = \mathfrak{P}_0 \supsetneq \mathfrak{P}_1 \supsetneq \cdots \supsetneq \mathfrak{P}_s$$

of length s such that $\mathfrak{P}_s \supseteq \mathfrak{p}B$. Since $\mathfrak{p} = \mathfrak{P} \cap A \supseteq \mathfrak{P}_i \cap A \supseteq \mathfrak{p}$, all $\mathfrak{P}_i$ lie over $\mathfrak{p}$. If $\mathrm{ht}(\mathfrak{p}) = r$, there exists $\mathfrak{p} \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r$ in A, and by going-down there exists a prime chain

$$\mathfrak{P}_s = \mathfrak{Q}_0 \supsetneq \mathfrak{Q}_1 \supsetneq \cdots \supsetneq \mathfrak{Q}_r$$

of B such that $\mathfrak{Q}_i \cap A = \mathfrak{p}_i$. Hence we obtain a prime chain of length $s+r$ in B, proving $\mathrm{ht}(\mathfrak{P}) \geqslant r+s$.

3. (a) follows from (2). For (b), take a minimal prime $\mathfrak{Q}$ over IB such that $\mathrm{ht}(\mathfrak{Q}) = \mathrm{ht}(IB)$, and put $\mathfrak{q} = \mathfrak{Q} \cap A$. Then $\mathrm{ht}(\mathfrak{Q}/\mathfrak{q}B) = 0$, and by (2) we have

$$\mathrm{ht}(IB) = \mathrm{ht}(\mathfrak{Q}) = \mathrm{ht}(\mathfrak{q}) \geqslant \mathrm{ht}(I)$$

Conversely, let $\mathfrak{q}$ be a minimal prime over I such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$, and take a prime $\mathfrak{Q}$ of B lying above $\mathfrak{q}$; we may assume $\mathfrak{Q}$ is minimal over $\mathfrak{q}B$. Then

$$\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{Q}) \geqslant \mathrm{ht}(IB)$$

□

3.7 Finitely Generated Extensions

Theorem 3.15. For a Noetherian ring A, we have

$$\dim A[x_1, \ldots, x_n] = \dim A + n$$

Proof. It suffices to show the case $n = 1$. Put $B = A[x]$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and $\mathfrak{P} \in \mathrm{Spec}(B)$ that is maximal among prime ideals lying over $\mathfrak{p}$. We contend $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}B) = 1$. We have $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = A[x] \otimes_A \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]$ so $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is a PID, and therefore every maximal prime has height one. Thus $\mathrm{ht}(\mathfrak{P}/\mathfrak{p}B) = 1$. Since $A[x]$ is free over A, Theorem 3.14 2. shows that $\mathrm{ht}(\mathfrak{P}) = \mathrm{ht}(\mathfrak{p}) + 1$. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, this gives $\dim B = \dim A + 1$. □

Definition. A ring A is **catenary** if for each $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(A)$,

(i) $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) < \infty$;

(ii) $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p})$.

- If A is Noetherian, (i) is automatically satisfied.
- For A Noetherian domain, TFAE:
  - A is catenary.
  - For each $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(A)$, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{q}/\mathfrak{p})$.
  - For each $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(A)$ with $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$, $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) + 1$.

- If A is catenary, so are its localizations and its quotient by an ideal.

A ring A is **universally catenary** if A is Noetherian and every A-algebra of finite type is catenary.

- A Noetherian ring A is universally catenary iff $A[x_1, \ldots, x_n]$ is catenary for all $n \geqslant 0$.
- If A is universally catenary, so are its localizations, quotient rings and any A-algebra of finite type.

Theorem 3.16. Let A be a Noetherian domain and $B \supseteq A$ an A-algebra of finite type that is an integral domain. Let $\mathfrak{P} \in \operatorname{Spec}(B)$ and $\mathfrak{p} = \mathfrak{P} \cap A$. Then

$$\operatorname{ht}(\mathfrak{P}) \leqslant \operatorname{ht}(\mathfrak{p}) + \operatorname{tr.deg}_A B - \operatorname{tr.deg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{P})$$

with equality when A is universally catenary, or if B is a polynomial ring of A. Here $\operatorname{tr.deg}_A B = \operatorname{tr.deg}_{\operatorname{Frac}(A)} \operatorname{Frac}(B)$.

Proof. Since transcendence degree is additive in tower of fields, by induction we may assume B is generated by single element, i.e. $B = A[x]$. Replacing A, B by the localization at $\mathfrak{p}$, we may assume $(A, \mathfrak{p})$ is local. Put $k = \kappa(\mathfrak{p}) = A/\mathfrak{p}$, and define

$$I = \{f(T) \in A[T] \mid f(x) = 0\}$$

Then $B = A[T]/I$. We divide the proof into two cases.

- $I = 0$. Then $B = A[T]$ is the polynomial ring, so that $\operatorname{tr.deg}_A B = 1$ and $B/\mathfrak{p}B = k[T]$. We have two cases.

  - $\mathfrak{P} \supsetneq \mathfrak{p}B$. Then $\operatorname{ht}(\mathfrak{P}/\mathfrak{p}B) = 1$. Notice that $B/\mathfrak{p}B = k[X]$ is a PID, and therefore $\mathfrak{P}/\mathfrak{p}B$ is maximal and principal, so $\kappa(\mathfrak{P}) = \dfrac{k[x]}{P/\mathfrak{p}\mathfrak{B}}$ is a finite extension of k. Thus $\operatorname{tr.deg}_k \kappa(\mathfrak{P}) = 0$.
  - $\mathfrak{P} = \mathfrak{p}B$. Then $\operatorname{ht}(\mathfrak{P}/\mathfrak{p}B) = 0$, and $\operatorname{tr.deg}_k \kappa(\mathfrak{P}) = 1$.

  In either case, we have $\operatorname{ht}(\mathfrak{P}/\mathfrak{p}B) = 1 - \operatorname{tr.deg}_k \kappa(\mathfrak{P})$. By Theorem 3.14, $\operatorname{ht}(\mathfrak{P}) = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{P}/\mathfrak{p}B)$. Combining these two equalities gives the result.

- $I \neq 0$. Then $\operatorname{tr.deg}_A B = 0$, since $\operatorname{Frac}(B) = K[T]/IK[T]$ is a finite extension of $K = \operatorname{Frac}(A)$. Denote by $\mathfrak{P}^*$ the inverse image of $\mathfrak{P}$ in $A[T]$; we have $\mathfrak{P} = \mathfrak{P}^*/I$ and $\kappa(\mathfrak{P}) = \kappa(\mathfrak{P}^*)$. Since A is a subring of $B = A[T]/I$, $A \cap I = 0$. Since $\operatorname{Spec}(K[T]) \to \operatorname{Spec}(A[T])$ is surjective and $A[T] \to K[T]$ is flat, by Theorem 3.14

  $$\operatorname{ht}(I) = \operatorname{ht}(IK[T]) \leqslant \dim K[x] = 1$$

  Since $I \neq 0$, $\operatorname{ht}(I) = 1$, and hence

  $$\operatorname{ht}(\mathfrak{P}) \leqslant \operatorname{ht}(\mathfrak{P}^*) - \operatorname{ht}(I) = \operatorname{ht}(\mathfrak{P}^*) - 1$$

  with equality if A is universally catenary (note that I is prime). On the other hand, we have $\operatorname{ht}(\mathfrak{P}^*) = \operatorname{ht}(\mathfrak{p}) + 1 - \operatorname{tr.deg}_k \kappa(\mathfrak{P}^*)$ by the first case, and $\kappa(\mathfrak{P}) = \kappa(\mathfrak{P}^*)$. These imply the result at once.

$\square$

4 Differentials

Definition. Let S be a ring and M an S-module. A map $d : S \to M$ is called a **derivation** if it satisfies the **Leibniz rule**, i.e., for all $f, g \in S$

$$d(fg) = fd(g) + gd(f)$$

If S is an R-algebra, we say d is **R-linear** if it's an R-homomorphism.

- The set $\mathrm{Der}_R(S, M)$ of all R-linear derivations $S \to M$ is naturally an S-module, given by

  $$bd : f \mapsto bd(f) \in M$$

  for all $b \in S$, $d \in \mathrm{Der}_R(S, M)$.

- For any derivation $d : S \to M$,

  $$d(1) = d(1 \cdot 1) = 1d(1) + 1d(1)$$

  so that $d(1) = 0$. It follows that d is R-linear iff $da = 0$ for all $a \in R$.

4.1 Kähler differentials

Definition. Let S be an R-algebra. The **module of Kähler differentials of S over R** is the S-module $\Omega_{S/R}$ together with an R-derivation $d : S \to \Omega_{S/R}$, called the **universal R-linear derivation**, that satisfies the following universal property:

$$\begin{aligned} \mathrm{Der}_R(S, M) &\xrightarrow{\sim} \mathrm{Hom}_S(\Omega_{S/R}, M) \\ e' : S \to M &\longmapsto e' \circ d : \Omega_{S/R} \to M \end{aligned}$$

is an isomorphism whenever M is an S-module.

- $\Omega_{S/R}$ exists and is unique up to a unique isomorphism. Precisely, we can take $\Omega_{S/R}$ to be the S-module generated by the symbol $\{df \mid f \in S\}$ subject to the Leibniz rules and R-linearity , and take $d : S \to \Omega_{S/R}$ to be the R-homomorphism defined by $d(f) = df$ for each $f \in S$.

- If $S = R[f_i]_{i \in I}$, then $\Omega_{S/R}$ is generated by df_i $(i \in I)$ as an S-module. In particular, $\Omega_{S/R}$ is finite over S if S is an R-algebra of finite type.

Proposition 4.1. If $S = R[x_1, \ldots, x_n]$ is a polynomial ring over R, then $\Omega_{S/R} = \bigoplus_{i=1}^{n} S dx_i$ is a free S-module. Explicitly, for $f \in S$, we have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Proof. Clearly $\Omega_{S/R}$ is generated by dx_i as S-modules, so we have a surjection $S^n \to \Omega_{S/R}$. On the other hand, the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$ defines a map $\Omega_{S/R} \to S$ by $\partial_i(dx_j) = \delta_{ij}$, so we have a map $(\partial_1, \dots, \partial_n) : \Omega_{S/R} \to S^n$. One can verify they are mutually inverse. $\square$

Example. Let R be a ring and S be a localization or a quotient of R. Then $\Omega_{S/R} = 0$.

- Say S is a quotient of R. Since $\Omega_{S/R}$ is generated by the symbols db for all $b \in S$ subject to the Leibniz rule and R-linearity, $dr = 0$ for all $r \in R$. For $b \in S$, say $r \in R$ is mapped to b; then $db = da = 0$.
- Say $S = T^{-1}R$ for some multiplicatively closed set $T \subseteq R$. For $s \in S$, there exists $t \in T$ with $ts \in R$. Then $0 = d(ts) = tds$, and hence $ds = 0$ for $t \in S$ is invertible.

We can view the assignment of Kähler differentials as a functor. Let $\mathcal{C}$ be the category whose object consists of all ring homomorphisms $\varphi : R \to S$. A morphism from $\varphi : R \to S$ to $\psi : R' \to S'$ is a pair of homomorphisms $f : R \to R'$, $g : S \to S'$ such that the diagram commutes:

$$\begin{tikzcd} S \arrow[r, "g"] & S' \\ R \arrow[u, "\varphi"] \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowleft"] & R' \arrow[u, "\psi"'] \end{tikzcd}$$

Define the functor $\Omega : \mathcal{C} \to \text{Mod}$ by assigning to each morphism $(R, S) \underset{\mathcal{C}}{\to} (R', S')$ the the morphism

$$\begin{tikzcd} \Omega_{S/R} \arrow[r] & \Omega_{S'/R'} \\ S \arrow[u, "d"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & S' \arrow[u, "d"'] \end{tikzcd}$$

where the bottom horizontal morphism is the given $S \to S'$, and the upper horizontal morphism is the unique S-module homomorphism induced from the universal property, by viewing $\Omega_{S'/R'}$ as an S-module.

In practice, the map $R \to R'$ will always be the identity, and the S-homomorphism $\Omega_{S/R} \to \Omega_{S'/R'}$ is replaced by the S'-homomorphism $S' \otimes_S \Omega_{S/R} \to \Omega_{S'/R'}$. We often call $\Omega_{S/R}$ the **relative cotangent functor**. It is, in the following sense, a right exact functor.

Proposition 4.2 (Relative cotangent sequence)**.** If $R \to S \to T$ are maps of rings, then we have an exact sequence of T-modules

$$\begin{tikzcd} T \otimes_S \Omega_{S/R} \arrow[r] & \Omega_{T/R} \arrow[r] & \Omega_{T/S} \arrow[r] & 0 \\ c \otimes db \arrow[r, mapsto] & cdb & & \\ & dc \arrow[r, mapsto] & dc & \end{tikzcd}$$

In addition, d has a left inverse if and only if any R-linear derivation $S \to M$ can be extended to a R-linear derivation $T \to M$ for all T-modules M.

Proof. Only the last statement needs a proof. d has a left inverse if and only if the induced map

$$\mathrm{Hom}_T(T \otimes_S \Omega_{S/R}, M) \leftarrow \mathrm{Hom}_T(\Omega_{T/R}, M)$$

is surjective for every T-module M, i.e. $\mathrm{Der}_R(S, M) \leftarrow \mathrm{Der}_R(T, M)$ is surjective. $\square$

Proposition 4.3 (Conormal sequence)**.** If $\pi : S \to T$ is a surjective R-algebra homomorphism with kernel I, then there is an exact sequence of T-modules

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{d} & T \otimes_S \Omega_{S/R} & \xrightarrow{D\pi} & \Omega_{T/R} & \longrightarrow & 0 \\ \overline{f} & \longmapsto & 1 \otimes df & & & & \\ & & c \otimes db & \longmapsto & cdb & & \end{array}$$

Moreover,

1. Put $S_1 := S/I^2$. Then $\Omega_{S/R} \otimes_R T \cong \Omega_{S_1/R} \otimes_{S_1} T$.

2. d has a left inverse iff $0 \to I/I^2 \to S_1 \to T \to 0$ splits.

Proof. Consider the map $d : I \to \Omega_{S/R}$ which is the restriction to I the universal derivation $S \to \Omega_{S/R}$.

- If $b \in S$, $c \in I$, $d(bc) = bd(c) + cd(b) \equiv bd(c) \pmod{I}$, so d induces an S-linear map $I \to \Omega_{S/R}/I\Omega_{S/R} = T \otimes_S \Omega_{S/R}$.

- Take $b \in I$ as well; this shows the induced map descends to a map $d : I/I^2 \to T \otimes_S \Omega_{S/R}$.

By the right adjointness of the tensor product, we see $T \otimes_S \Omega_{S/R}$ is generated as a T-module by db for $b \in S$ subject to the Leibniz rules and R-linearity. This is the same as the description of $\Omega_{T/R}$, except that the elements df with $f \in I$ are replaced by $d0 = 0$ (for $0 \in R$). Thus $\Omega_{T/R}$ is the cokernel of $d : I/I^2 \to T \otimes_S \Omega_{S/R}$ as claimed.

1. It's an isomorphism if and only if for each T-module M,

   $$\mathrm{Hom}_T(\Omega_{S/R} \otimes_S T, M) \leftarrow \mathrm{Hom}_T(\Omega_{S_1/R} \otimes_{S_1} T, M)$$

   is an isomorphism, i.e. $\mathrm{Der}_R(S, M) \leftarrow \mathrm{Der}_R(S_1, M)$ is an isomorphism for each S/I-module M. This is clear from the computation in the beginning of the proof.

2. By 1., we can replace S and I by S/I^2 and I/I^2, respectively, so that we may assume $I^2 = 0$. Suppose d has a left inverse $\sigma : T \otimes_S \Omega_{S/R} \to I$. Putting $Db := \sigma(1 \otimes db)$ for each $b \in S$, we defines an R-linear derivation $D : S \to I$ such that $Df = f$ for each $f \in I$. Then the map $\tau : S \to S$ defined by $\tau = \mathrm{id}_S - D$ is an R-algebra homomorphism vanishing on I, so it induces a map $\tau : T \cong S/I \to S$. Now $\pi\tau = \pi(\mathrm{id}_S - D) = \pi$, for π vanishes on I, proving that the sequence splits.

Conversely, let $\tau : T \to S$ be a right inverse of $\pi : S \to T \cong S/I$. Define $D : S \to S$ by $D = \mathrm{id}_S - \tau\pi$. Define $\sigma : \Omega_{S/R} \to S$ by $\sigma(db) = D(b)$; since $\pi\sigma(db) = \pi D(b) = \pi(b) - \pi(b) = 0$, we see the image of σ is contained in I. Since $I^2 = 0$, I is a T-module, so we can extend σ to a T-homomorphism $\sigma : T \otimes_S \Omega_{S/R} \to I$. Finally, for each $f \in I$, $\sigma(df) = D(f) = f - \tau\pi(f) = f = \mathrm{id}_I(f)$, so σ is a left inverse of d. $\square$

Corollary 4.3.1 (Coequalizer)**.** If $T = \mathrm{coequal}(\psi, \psi' : S_1 \to S_2)$ is the coequalizer in the category of R-algebras, then there is an exact sequence of T-modules

$$T \otimes_{S_1} \Omega_{S_1/R} \xrightarrow{\mathrm{id}_T \otimes D\psi - \mathrm{id}_T \otimes D\psi'} T \otimes_{S_2} \Omega_{S_2/R} \longrightarrow \Omega_{T/R} \longrightarrow 0$$

Proof. By the conormal sequence, $\Omega_{T/R}$ is the quotient of $T \otimes_{S_2} \Omega_{S_2/R}$ by the submodule generated by the elements $1 \otimes d(\psi(b) - \psi'(b))$. This submodules is precisely the image of $\mathrm{id}_T \otimes D\psi - \mathrm{id}_T \otimes D\psi'$. $\square$

Example. If S is of finite type over R, say $S = R[x_1, \ldots, x_n]/I$ with $I = (f_1, \ldots, f_m)$, then

- $S \otimes_R \Omega_{R[x_1,\ldots,x_n]/R} = \bigoplus_{i=1}^{n} S dx_i$.
- By the conormal sequence,
$$\Omega_{S/R} = \mathrm{coker}\left(d : I/I^2 \to \bigoplus_{i=1}^{n} S dx_i\right)$$
- Writing I/I^2 as a homomorphic image of $\bigoplus_{i=1}^{m} S e_i$, where e_i is sent to f_i. Then the composition
$$\mathcal{J} : \bigoplus_{i=1}^{m} S e_i \longrightarrow\!\!\!\!\rightarrow I/I^2 \longrightarrow \bigoplus_{i=1}^{n} S dx_i$$
is a map of free S-modules whose matrix representation is the Jacobi matrix of the f_j with respect to the x_i; the (i, j) entry of $\mathcal{J}$ is $\partial f_j / \partial x_i$.

Hence, $\Omega_{S/R}$ is the cokernel of the Jacobi matrix $\mathcal{J} = (\partial f_j / \partial x_i)$.

For an explicit example, consider the ring $S = R[x, y, t]/(y^2 - x^2(t^2 - x))$. In the case we have

$$\mathcal{J} = \begin{pmatrix} 3x^2 - 2xt^2 \\ 2y \\ -2x^2 t \end{pmatrix}$$

and following the computation above we see $\Omega_{S/R}$ is the S-module generated by dx, dy, dt with a single relation

$$(3x^2 - 2xt^2)dx + (2y)dy - (2x^2 t)dt = 0$$

Proposition 4.4 (Base change)**.** For any R-algebras R' and S, we have the commutative diagram

$$\begin{tikzcd} & R' \otimes_R \Omega_{S/R} \arrow[dd, "\wr", no head] \\ R' \otimes_R S \arrow[ur, "1\otimes d"] \arrow[dr, "d"'] & \\ & \Omega_{(R'\otimes_R S)/R'} \end{tikzcd}$$

Proof. From the morphism $(R, S) \to (R', R' \otimes_R S)$, we obtain a map $R' \otimes_R \Omega_{S/R} \to \Omega_{(R'\otimes_R S)/R'}$, sending $a' \otimes db$ to $d(a' \otimes b)$. On the other hand, $1 \otimes d$ is an R'-linear derivation, so from the universal property we obtain $\Omega_{(R'\otimes_R S)/R'} \to R' \otimes_R \Omega_{S/R}$, sending $d(a' \otimes b)$ to $a' \otimes db$. □

Proposition 4.5 (Tensor product)**.** If $T = \bigotimes_R S_i$ is the tensor product (coproduct) of some R-algebras S_i, then

$$\Omega_{T/R} \cong \bigoplus\nolimits_i (T \otimes_{S_i} \Omega_{S_i/R}) = \bigoplus\nolimits_i \big((\bigotimes\nolimits_{R, j\neq i} S_j) \otimes_R \Omega_{S_i/R}\big)$$

by an isomorphism α satisfying

$$\alpha : d(\cdots \otimes 1 \otimes b_i \otimes 1 \otimes \cdots) \mapsto (\ldots, 0, 1 \otimes db_i, 0, \ldots)$$

where $b_i \in S_i$ occurs in the i-th place in each expression.

Proof. The second equality is clear. Denote by Ω the middle object. Write $d_i : S_i \to \Omega_{S_i/R}$ for the universal derivation. Then we have

$$1 \otimes d_i : T = \Big(\bigotimes\nolimits_{j\neq i} S_j\Big) \otimes_R S_i \to \Big(\bigotimes\nolimits_{j\neq i} S_j\Big) \otimes_R \Omega_{S_i/R}$$

Only finitely many of the maps $1 \otimes d_i$ are nonzero on a given element in T, so the map $e : T \to \Omega$ given by $e = \sum_i 1 \otimes d_i$ is well-defined. Since e is a sum of derivation, it is a derivation itself, so it gives a map $\alpha : \Omega_{T/R} \to \Omega$ defined by $d(\otimes_i b_i) \mapsto e(\otimes_i b_i)$.

Conversely, for each S_i consider the composition $S_i \to T \to \Omega_{T/R}$; by the universal property, it gives

$$\begin{aligned} \beta_i : T \otimes_{S_i} \Omega_{S_i/R} &\longrightarrow \Omega_{T/R} \\ 1 \otimes d_i b_i &\longmapsto d(1 \otimes b_i) \end{aligned}$$

All β_i then produce a map $\Omega \to \Omega_{T/R}$, and it's inverse to α. □

Corollary 4.5.1. If $T = S[x_1, \ldots, x_n]$ is a polynomial ring over an R-algebra S, then

$$\Omega_{T/R} \cong (T \otimes_S \Omega_{S/R}) \oplus \left(\bigoplus_{i=1}^{n} T dx_i \right)$$

Proof. Put $T' = R[x_1, \ldots, x_n]$; then $T = S \otimes_R T'$. By Proposition,

$$\Omega_{T/R} \cong (T \otimes_S \Omega_{S/R}) \oplus (T \otimes_{T'} \Omega_{T'/R})$$

The final expression results from Proposition 4.1. $\square$

Example. Let R, S, T be as in the corollary. Let I be an ideal of T and form $T' = T/I$. From the conormal sequence we have an exact sequence

$$I/I^2 \xrightarrow{d} T' \otimes \Omega_{T/R} \longrightarrow \Omega_{T'/R} \longrightarrow 0$$

with d defined by $d(\overline{f}) = 1 \otimes d_{T/R} f$. The isomorphism in the corollary is given by

$$\Omega_{T/R} \xrightarrow{\qquad\qquad\qquad} (T \otimes_S \Omega_{S/R}) \oplus \left(\bigoplus_{i=1}^n T dx_i\right)$$

$$d_{T/R}(a x_1^{r_1} \cdots x_n^{r_n}) \longmapsto x_1^{r_1} \cdots x_n^{r_n} . d_{S/R} a + \sum_{i=1}^n a \frac{\partial (x_1^{r_1} \cdots x_n^{r_n})}{\partial x_i} dx_i$$

Combining these two isomorphisms we obtain

$$\Omega_{T'/R} = \frac{(T' \otimes_S \Omega_{S/R}) \oplus \left(\bigoplus_{i=1}^n T' dx_i\right)}{\left\langle (dP)(x) + \sum_{i=1}^n \frac{\partial P}{\partial x_i} dx_i \,\middle|\, P \in I \right\rangle_{T'}}$$

where $(dP)(x) \in T \otimes_S \Omega_{S/R}$ is a polynomial obtained by applying $d_{S/R}$ to each coefficient of P.

Theorem 4.6 (Colimits)**.** Let $\mathcal{B}$ be a diagram in the category of R-algebras, and set $T := \varinjlim \mathcal{B}$. If F is the functor from $\mathcal{B}$ (identifying with its image) to the category of T-modules taking an object S to $T \otimes_S \Omega_{S/R}$ and a morphism $\varphi : S' \to S$ to the morphism $1 \otimes D\varphi : T \otimes_S (S \otimes_{S'} \Omega_{S'/R}) \to T \otimes_S \Omega_{S/R}$, then

$$\Omega_{T/R} = \varinjlim F$$

Proposition 4.7 (Localization)**.** If S is an R-algebra and $T \subseteq S$ is a multiplicatively closed set, then

$$\Omega_{T^{-1}S/R} \cong T^{-1} S \otimes_S \Omega_{S/R}$$

in such a way that $d(1/t) = -t^{-2} dt$ for all $t \in T$.

Proof.

Method 1 First suppose $T = \{t^n \mid n \geqslant 1\}$ for a single $t \in S$. Then $T^{-1}S \cong S[x]/(tx - 1)$. By Corollary 4.5.1 and conormal sequence

$$\Omega_{T^{-1}S/R} \cong \frac{T^{-1} S \Omega_{S/R} \oplus T^{-1} S dx}{T^{-1} S d(tx - 1)} = \frac{T^{-1} S \Omega_{S/R} \oplus T^{-1} S dx}{T S^{-1} (t dx + x dt)}$$

Since $t \in T^{-1}S$ is invertible, we see $\Omega_{T^{-1}S/R} = T^{-1} S \Omega_{S/R}$ with dx identified with $-\dfrac{x}{t} dt$. Thinking of x as t^{-1}, this reads $d(1/t) = -t^{-2} dt$.

For the general case, recall that $T^{-1}S = \varinjlim_{t \in T} S_t$. Hence by Theorem above, we hae

$$\Omega_{T^{-1}S/R} = \varinjlim_{t \in T} T^{-1} S \otimes_{S_t} \Omega_{S_t/R} = \varinjlim_{t \in T} T^{-1} S \otimes_{S_t} S_t \otimes_S \Omega_{S/R} = T^{-1} S \otimes_S \Omega_{S/R}$$

Method 2 We prove there exists $d' : T^{-1}S \to T^{-1}S \otimes_S \Omega_{S/R}$ sending $1/t$ to $-t^{-2}dt$ satisfies a commutative diagram

$$\begin{tikzcd} & T^{-1}S \otimes_S \Omega_{S/R} \arrow[dd, dash, "\wr"] \\ T^{-1}S \arrow[ur, "d'"] \arrow[dr, "d"'] & \\ & \Omega_{T^{-1}S/R} \end{tikzcd}$$

First, the composition $S \to T^{-1}S \to \Omega_{T^{-1}S/R}$ is an R-linear derivation, and hence it gives an S-homomorphism $\Omega_{S/R} \to \Omega_{T^{-1}S/R}$, or equivalently, a $T^{-1}S$-homomorphism $T^{-1}S \otimes_S \Omega_{S/R} \to \Omega_{T^{-1}S/R}$, mapping $bt^{-1} \otimes ds$ to $bt^{-1}ds$.

Second, to get a upward map, we must verify that d' is a well-defined R-linear derivation. Suppose $b/s = 0$ in $T^{-1}S$; then $bt = 0$ for some $s \in T$. Then $d(b/s) = s^{-2}(sbd - bds)$. Since

$$t^2(sbd - bds) = ts(tbd + bdt) - s(bt)dt - t(bt)ds = 0$$

we see $d(b/s) = 0$ in $T^{-1}S \otimes_S \Omega_{S/R}$, so that d' is well-defined. It's clear a derivation , as one learnt in calculus. Hence, by the universal property, we obtain a map $\Omega_{T^{-1}S/R} \to T^{-1}S \otimes_S \Omega_{S/R}$ carrying $cd(b/s)$ to $-cs^{-2} \otimes (sbd - bds)$. In this stage it is direct to see the obtained maps are mutually inverse.

Method 3 Use the relative cotangent sequence and the example preceding it. We must show $\mathrm{Der}_R(S, M) \leftarrow \mathrm{Der}_R(T^{-1}S, M)$ is surjective for each $T^{-1}S$-module M. Let $D : S \to M$ be an R-linear derivation, and for $b/s \in T^{-1}S$ with $b \in S$, $s \in T$, define $\overline{D}(b/s) = \dfrac{sD(b) - bD(s)}{s^2}$. We must show it defines a well-defined derivation $\overline{D} : T^{-1}S \to M$ whose restriction to S is D. This is already shown in Method 2. $\square$

Proposition 4.8 (Finite direct products)**.** If $S_1, \ldots, S_n$ are R-algebras and $S = \prod_i S_i$, then

$$\Omega_{S/R} = \prod_{i=1}^{n} \Omega_{S_i/R}$$

Proof. If e_i is the idempotent of S that is the unit of S_i, and $D \in \mathrm{Der}_R(S, M)$ for some S-module M, then $(2e_i - 1)D(e_i) = 0$. Since $(2e_i - 1)^2 = 4e_2^2 - 4e_i + 1 = 1$, $2e_i - 1$ is a unit, and hence $De_i = 0$. Therefore, $D(e_i f) = e_i Df$ for all $f \in S$. Consequently, D maps $S_i := e_iS$ to $M_i = e_iM$, and corresponds to a unique map $\Omega_{S_i/R} \to M_i$. It follows that $S \to \prod_i \Omega_{S_i/R}$ satisfies the universal property of $\Omega_{S/R}$. $\square$

4.2 Kähler differential as a conormal module

Lemma 4.9. Let $\varphi : S \to S'$ be a map of R-algebras, and let $\delta : S \to S'$ be a map of abelian groups. If $\delta(S)^2 = 0$, then $\varphi + \delta$ is a homomorphism of R-algebras if and only if δ is an R-linear derivation, in the sense that

$$\delta(b_1b_2) = \varphi(b_1)\delta(b_2) + \varphi(b_2)\delta(b_1)$$

Proof. By computation,

$$(\varphi+\delta)(b_1b_2) = \varphi(b_1b_2) + \delta(b_1b_2)$$
$$(\varphi+\delta)(b_1)\cdot(\varphi+\delta)(b_2) = \varphi(b_1b_2) + \varphi(b_1)\delta(b_2) + \varphi(b_2)\delta(b_1) + \delta(b_1)\delta(b_2)$$

The last term in the second identity is zero, so they are equal if and only if

$$\delta(b_1b_2) = \varphi(b_1)\delta(b_2) + \varphi(b_2)\delta(b_1)$$

Also, for $r \in R$, $s \in S$, we have $(\varphi+\delta)(rs) = r(\varphi+\delta)(s)$ if and only if $\delta(rs) = r\delta(s)$. □

Let $R \to S$ be a ring homomorphism. Consider the multiplication $\mu : S \otimes_R S \to S$, and denote $I = \ker\mu$. Let $e : S \to I/I^2$ be the map induced by $b \mapsto 1 \otimes b - b \otimes 1$.

Theorem 4.10. $(e, I/I^2)$ is naturally isomorphic to $(d, \Omega_{S/R})$.

Proof. We first show that $e : S \to I/I^2$ is a derivation. Consider the exact sequence

$$I/I^2 \longrightarrow (S \otimes_R S)/I^2 \longrightarrow S \longrightarrow 0$$

First, note that S acts on I/I^2 naturally by any section of $(S \otimes_R S)/I^2 \to S$. Note that $b \mapsto 1 \otimes b$ and $b \mapsto b \otimes 1$ are two homomorphic sections of $S \to (S \otimes_R S)/I^2$, so $e(T) \subseteq I/I^2$. By Lemma, e is an R-derivation.

By the universal property for $(d, \Omega_{S/R})$, there exists a unique map $\varphi : \Omega_{S/R} \to I/I^2$ such that $e = \varphi d$. We shall prove φ is an isomorphism.

Let $T = S \oplus \Omega_{S/R}$ be the abelian group direct sum. For $b, b' \in S$ and $u, u' \in \Omega_{S/R}$, define

$$(b,u)(b',u') := (bb', bu' + b'u)$$

This is the trivial extension of S by $\Omega_{S/R}$; S acts on T as $S \oplus \{0\}$. Define the ring homomorphism

$$\psi : S \otimes_R S \longrightarrow T$$
$$a \otimes b \longmapsto (ab, adb)$$

by $\psi_1 : S \ni b \mapsto (b, db) \in T$ and $\psi_2 : S \ni a \mapsto (a, 0) \in T$; ψ_2 is clearly a ring homomorphism, and since

$$\psi_1(ab) = (ab, d(ab)) = (ab, adb + bda) = (a, da)(b, db) = \psi_1(a)\psi_1(b)$$

ψ_1 is also a ring homomorphism. Then by construction, $\psi(I) \subseteq \{0\} \oplus \Omega_{S/R}$; let $\psi' : I \to \Omega_{S/R}$ given by $\psi(a) = (0, \psi'(a))$ for all $a \in I$. Since ψ is a ring homomorphism, according to the multiplication rule on T, $\psi(I^2) = 0$, and hence ψ' descends to a map $\psi' : I/I^2 \to \Omega_{S/R}$. Finally,

- For $b \in S$,
$$\psi'\varphi(db) = \psi'(1 \otimes b - b \otimes 1) = db$$
- For $x = \sum a_i \otimes b_i \in I/I^2$,
$$\varphi\psi'(\sum a_i \otimes b_i) = \varphi(\sum a_i db_i) = \sum a_i(1 \otimes b_i - b_i \otimes 1) = x - (\sum a_i b_i) \otimes 1 = x$$

□

4.3 Field extensions

4.3.1 Separable generation

Definition. Let K/k be a field extension.

1. A transcendence basis $(x_\lambda)_\lambda$ of K/k is called a **separating transcendence basis** if K is separably algebraic over $k(x_\lambda)_\lambda$.

2. K is **separably generated** over k if K/k has a separating transcendence basis.

Put $r(K) := \operatorname{rank}_K \Omega_{K/k}$. Suppose $L = K(t)$. We compare $r(K)$ and $r(L)$.

Case 1. t is transcendental over K. Then by Corollary 4.5.1,

$$\Omega_{K[t]/k} = (K[t] \otimes_K \Omega_{K/k}) \oplus (K[t]dt)$$

and by Proposition 4.7

$$\Omega_{L/k} = (L \otimes_K \Omega_{K/k}) \oplus Ldt$$

Thus $r(L) = r(K) + 1$.

Case 2. t is separably algebraic over K. Then $L = K[t] = K[X]/(f(X))$ with $f = m_{t,K}$. By Example, we see

$$\Omega_{L/k} = \frac{(L \otimes_K \Omega_{K/k}) \oplus LdX}{\langle (df)(t) + f'(t)dX \rangle_L}$$

Since $f'(t) \neq 0$ is invertible in L, $\Omega_{L/k} \cong L \otimes_K \Omega_{K/k}$, so thus $r(L) = r(K)$. From this we see any derivation of K into L can be extended uniquely to a derivation of L.

Case 3. $\operatorname{Char}(k) = p$, $t^p = a \in K$, $t \notin K$, $d_{K/k}a = 0$. Then $L = K[t] = K[X]/(X^p - a)$. Consider the isomorphism in Case 2 with $f(X) = X^p - a$; since $(df)(t) + f'(t)dX = 0$, we have

$$\Omega_{L/k} = (L \otimes_K \Omega_{K/k}) \oplus LdX$$

so $r(L) = r(K) + 1$.

Case 4. $\operatorname{Char}(k) = p$, $t^p = a \in K$, $t \notin K$, $d_{K/k}a \neq 0$. Again, but $(df)(t) + f'(t)dX = -d_{K/k}a \neq 0$, so $r(L) = r(K)$.

Theorem 4.11.

1. Let k be a field, K/k an extension and L/K a finitely generated extension. Then

$$\operatorname{rank}_L \Omega_{L/k} \geqslant \operatorname{rank}_K \Omega_{K/k} + \operatorname{tr.deg}_K L$$

with equality if L/K is separably generated.

2. Let L/k be a finitely generated extension. Then

$$\operatorname{rank}_L \Omega_{L/k} \geqslant \operatorname{tr.deg}_K L$$

with equality if and only if L/k is separably generated over k. In particular, $\Omega_{L/k} = 0$ if and only if L/K is separably algebraic over k.

Proof.

1. This follows from the above discussion.

2. The inequality is a special case of 1. Now suppose $\Omega_{L/k} = 0$, i.e., $r(L) = 0$. Then for $k \subseteq K \subseteq L$ we have $r(K) = 0$, so the case 1, 3, 4 above cannot occur, implying L/k is separably algebraic. Suppose next that $r(L) = \operatorname{tr.deg}_k L = r$. Let $x_1, \ldots, x_r \in L$ such that $dx_1, \ldots, dx_r$ form a basis for $\Omega_{L/k}$ over L. The relative cotangent sequence

$$L \otimes_{k(x_1,\ldots,x_r)} \Omega_{k(x_1,\ldots,x_r)/k} \to \Omega_{L/k} \to \Omega_{L/k(x_1,\ldots,x_r)} \to 0$$

implies $\Omega_{L/k(x_1,\ldots,x_r)} = 0$, so $L/k(x_1,\ldots,x_r)$ is separably algebraic as shown above. Since $r = \operatorname{tr.deg}_k L$, the elements $x_1, \ldots, x_r$ form a transcendence basis for L/k.

□

Lemma 4.12. Let K/k be an algebraic extension. TFAE:

(1) K/k is separably algebraic;

(2) The ring $K \otimes_k k'$ is reduced for any extension k'/k;

(3) ditto for any algebraic extension k'/k;

(4) ditto for any finite extension k'/k.

Proof. Each property holds if and only if it holds for any finite subextension of K/k, so we may assume that $[K : k] < \infty$.

(1) $\Rightarrow$ (2): Suppose K/k is finite separable. Then $K = k(t)$ for some $t \in K$ by the primitive element theorem. Let $f(X) = m_{t,k}(X)$ be the minimal polynomial of t; then $K \cong k[X]/(f(X))$, and thus $K \otimes_k k' \cong k'[X]/(f(X))$. Since $f(X)$ is separable, by Chinese Remainder theorem we see $K \otimes_k k'$ becomes a direct product of finite separable extensions of k', and thus $K \otimes_k k'$ is reduced.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Suppose $\operatorname{Char} k = p > 0$ and K contains an inseparable element t over k. Let $f(X) = m_{t,k}(X)$; then $f(X) = g(X^p)$ for some $g \in k[X]$. Say $g(X) = a_0 + a_1 X \cdots + a_n X^n$ and put

$$k' := k(a_0^{\frac{1}{p}}, \ldots, a_n^{\frac{1}{p}})$$

Then $f(X) = g(X^p) = h(X)^p$ for some $h(X) \in k'[X]$, and $k(t) \otimes_k k' = k'[X]/(h(X)^p)$ has nilpotent elements. Since k' is a field, $k(t) \otimes_k k'$ is a subring of $K \otimes_k k'$, so that the condition (4) does not hold.

$\square$

4.3.2 Separable algebra

Definition. Let k be a field and A a k-algebra. We say A is **separable** over k if for any algebraic extension k'/k, the ring $A \otimes_k k'$ is reduced.

There are some immediate consequences of the definition.

- If A is separable, then so is any k-subalgebra of A.
- If all finitely generated k-subalgebras of A are separable, then so is A.
- If for any finite extension k'/k, the ring $A \otimes_k k'$ is reduced, then A is separable.

  Proof. Suppose for some algebraic extension k'/k, the ring $A \otimes_k k'$ contains a nilpotent, say $t \neq 0$ and $t^\ell = 0$. Write $t = \sum_{i=1}^{n} a_i \otimes c_i$ and $t^\ell = \sum_{i=1}^{m} (a'_i, c'_i)$ in $A \times k'$ for some $a_i, a'_i \in A$, $c_i, c'_i \in k'$ and form the field $k'' := k(c_i, c'_j)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}}$. Then $0 \neq t \in A \otimes_k k''$ and $t^\ell = 0$ in $A \otimes_k k''$. Thus $A \otimes_k k''$ is not reduced. $\square$

Lemma 4.13. If k'/k is a separately generated extension, and if A is a reduced k-algebra, then $A \otimes_k k'$ is reduced.

Proof. We may assume A is finitely generated over k. Since A is reduced, the homomorphism

$$A \longrightarrow A' := \prod_{\mathfrak{p}\text{: minimal}} A_{\mathfrak{p}}$$

is injective, and each $A_{\mathfrak{p}}$ is a field. Since A is Noetherian, the RHS is actually a finite product. (Actually, A' is isomorphic to $\operatorname{Frac}(A)$.) Since $A \otimes_k k' \subseteq A' \otimes_k k'$, we may further assume A is a field.

It suffices to consider the cases k'/k is separably algebraic and k'/k is purely transcendental. Then the former case follows from Lemma 4.12. In the latter case, say $k' = k(t_1, \ldots, t_n)$, then $A \otimes_k k' \subseteq A(t_1, \ldots, t_n)$ is reduced. $\square$

Corollary 4.13.1. If k is a perfect field, then a k-algebra A is separable if and only if it is reduced. In particular, any extension field K of k is separable over k (as k-algebras).

Proof. k being perfect, any algebraic extension k' of k is separable, so $A \otimes_k k'$ is reduced by lemma above, Conversely, if A is separable, then by definition $A = A \otimes_k k$ is reduced. $\square$

Lemma 4.14. Let k be a field with $\operatorname{Char} k = p > 0$, and K/k a finitely generated field extension. TFAE:

(i) K is separable over k (as k-algebras);

(ii) the ring $K \otimes_k k^{\frac{1}{p}}$ is reduced;

(iii) K is separably generated over k.

where $k^{\frac{1}{p}} := \{y \in \overline{k} \mid y^p \in k\}$ and $\overline{k}$ is the algebraic closure of k.

Proof.

(iii) $\Rightarrow$ (i) If K/k is separably generated, then $k' \otimes_k K$ is reduced for any extension k'/k by Lemma 4.13.

(i) $\Rightarrow$ (ii) Trivial.

(ii) $\Rightarrow$ (iii) Let $K = k(x_1, \ldots, x_n)$; suppose $x_1, \ldots, x_r$ form a transcendence basis for K/k, and suppose $x_{r+1}, \ldots, x_q$ are separable over $k(x_1, \ldots, x_r)$ while x_{q+1} is not. Put $y = x_{q+1}$ and $f(Y^p)$ be the minimal polynomial of y over $k(x_1, \ldots, x_r)$. Clearing the denominators of the coefficients of f we obtain a polynomial $F(X_1, \ldots, X_r, Y^p)$ irreducible in $k[X, Y]$ such that $F(x_1, \ldots, x_r, y^p) = 0$. Then there must be at least one X_i such that $\dfrac{\partial F}{\partial X_i} \neq 0$, for otherwise ($F$ is a polynomial in X^p) we would has $F(X, Y^p) = G(X, Y)^p$ with $G = k^{\frac{1}{p}}[X, Y]$ so that

$$k[x_1, \ldots, x_r, y] \otimes_k k^{\frac{1}{p}} \cong \frac{k[X, Y]}{(F(X, Y^p))} \otimes_k k^{\frac{1}{p}} \cong \frac{k^{\frac{1}{p}}[X, Y]}{(G(X, Y)^p)}$$

has a nilponent element; since $k[x_1, \ldots, x_r, y] \otimes_k k^{\frac{1}{p}}$ is a subring of $K \otimes_k k^{\frac{1}{p}}$, this leads to a contradiction to (ii). Hence, say $\dfrac{\partial F}{\partial X_1} \neq 0$, so that x_1 is separably algebraic over $k(x_2, \ldots, x_r, y)$, and the same holds for $x_{r+1}, \ldots, x_q$ as well. Exchanging x_1 with $y = x_{q+1}$ we have that $x_{r+1}, \ldots, x_{q+1}$ are separable over $k(x_1, \ldots, x_r)$. Now by induction on $q \geqslant r + 1$ we see that we can choose a separating transcendence basis for K/k from the set $\{x_1, \ldots, x_n\}$.

- (Exchanging x_1 and y is valid) Consider the lattice

$$\begin{tikzcd}
 & k(x_1, \dots, , x_r, y) \arrow[dl, dash, "\text{finite sep}"'] \arrow[dr, dash, "\text{finite}"] & \\
k(x_2, \dots, x_r, y) \arrow[dr, dash] & & k(x_1, \dots, x_r) \arrow[dl, dash] \\
 & k(x_2, \dots, x_r) \arrow[d, dash] & \\
 & k &
\end{tikzcd}$$

  For x_i ($1 \leqslant i \leqslant n$), since x_i is algebraic over $k(x_1, \dots, x_r)$, it is also algebraic over $k(x_1, \dots, x_r, y)$. Since $k(x_1, \dots, x_r, y)/k(x_2, \dots, x_r, y)$ is finite, x_i is also algebraic over $k(x_2, \dots, x_r, y)$. This means K is algebraic over $k(x_2, \dots, x_r, y)$; since $\operatorname{tr.deg}_k K = r$, $x_2, \dots, x_r, y$ are algebraically independent over k.

- We know $x_{r+1}, \dots, x_q$ are separably algebraic over $k(x_1, \dots, x_r)$. Let $r + 1 \leqslant i \leqslant q$. Since $k(x_1, \dots, x_r, y)/k(x_2, \dots, x_r, y)$ is finite separable, that x_i is separable over $k(x_2, \dots, x_r, y)$ is equivalent of that x_i is separable over $k(x_1, x_2, \dots, x_r, y)$, the latter being true for x_i is separable over $k(x_1, \dots, x_r)$.

□

Proposition 4.15. Let k be a field and A a separable k-algebra. Then for any extension k' of k (algebraic or not), the ring $A \otimes_k k'$ is reduced and is a separable k'-algebra.

Proof. Enough to prove that $A \otimes_k k'$ is reduced. The statement holds for any algebraic extension of k (by definition of a separable algebra), so we may assume k' contains the algebraic closure $\overline{k}$ of k. (This is to deal with the case $\operatorname{Char} k > 0$.)

Since $A \otimes_k \overline{k}$ is reduced by assumption, and since any finitely generated extension of $\overline{k}$ is separably generated (Lemma 4.13.1 for $\operatorname{Char} k = 0$ and Lemma 4.14.(ii) for $\operatorname{Char} k > 0$), the ring $A \otimes_k k' = (A \otimes_k \overline{k}) \otimes_{\overline{k}} k'$ is reduced by Lemma 4.13. □

4.3.3 Linear disjointness

Definition. Let $k \subseteq L$ be a field and K, K' be two subfields of L containing k. We say K, K' are **linearly disjoint over** k if they satisfies the following equivalent conditions.

(a) If $\alpha_1, \dots, \alpha_n \in K$ are k-linearly independent, they are K'-linearly independent.

(b) The condition (a) holds if we interchange K and K'.

(c) The canonical homomorphism $K \otimes_k K' \to KK' \subseteq L$ is an isomorphism.

Proof. It suffices to show the equivalence (a)$\Leftrightarrow$(c).

(a)$\Rightarrow$(c): Say $\xi = \sum_{i=1}^{n} x_i \otimes y_i$ lies in the kernel of the homomorphism. Assume $x_1, \ldots, x_r$ are maximal k-linearly independent among the set $\{x_1, \ldots, x_n\}$; then we can write $\xi = \sum_{i=1}^{r} x_i \otimes y_i'$. If $\xi \neq 0$, then $y_i' \neq 0$ for some i, while its image $\sum_{i=1}^{r} x_i y_i' = 0 \in KK'$ a contradiction to (a).

(c)$\Rightarrow$(a): (c) implies that if $\alpha_1, \ldots, \alpha_n \in K$ are k-linearly independent and $\beta_1, \ldots, \beta_m \in K'$ are k-linearly independent, then the $\alpha_i\beta_j$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$ are k-linearly independent. Indeed, the $\alpha_i \otimes \beta_j \in K \otimes_k K'$ are k-linearly independent, so the isomorphism implies so are the $\alpha_i\beta_j$. Now (a) is clear.

$\square$

Theorem 4.16 (MacLane)**.** Let k be a field with $\operatorname{Char} k = p > 0$ and K/k an extension. TFAE:

(a) K is separable over k (as k-algebras);

(b) K and $k^{p^{-\infty}} := \{x \in \overline{k} \mid x^{p^n} \in k \text{ for some } n \geqslant 1\}$ are linearly disjoint over k.

(c) K and $k^{\frac{1}{p}}$ are linearly disjoint over k.

Proof.

(a)$\Rightarrow$(b) Suppose $\alpha_1, \ldots, \alpha_n \in K$ are k-linearly independent and $\sum_{i=1}^{n} c_i\alpha_i = 0$ for some $c_i \in k^{p^{-\infty}}$. Let $k' := k(c_1, \ldots, c_n)$; the $k'^{p^\ell} \subseteq k$ for some $\ell \gg 0$, and $A := K \otimes_k k'$ is reduced. Since A is a finite dimensional K-algebras, A is Artinian, and thus every prime ideal is maximal. Let $\mathfrak{m}_i \in \operatorname{Spec}(A)$ $(i = 1, 2)$; then $\mathfrak{m}_i^{p^\ell} \subseteq K$ so $\mathfrak{m}_i^{p^\ell} = 0$; taking radical, one gets $\mathfrak{m}_1 = \mathfrak{m}_2$. Thus A has only one prime ideal, and since A is reduced, it follows that A is a field, forcing that $A = K[k']$. Thus $\sum_{i=1}^{n} \alpha_i \otimes c_i = 0$ so that $c_i = 0$.

(b)$\Rightarrow$(c) Trivial.

(c)$\Rightarrow$(a) Since then $K \otimes_k k^{\frac{1}{p}}$ is a field, it is reduced, and thus K is separable over k by Lemma 4.14.

$\square$

5 Formal Smoothness

6 Spectral Sequences

Let $\mathcal{C}$ be an abelian category, and $\mathrm{Kom}(\mathcal{C})$ be the category of complexes in $\mathcal{C}$. Let $(K^\bullet, d^\bullet) \in \mathrm{Kom}(\mathcal{C})$ and $F^pK^\bullet$ $(p \in \mathbb{Z})$ a decreasing filtration of $K^\bullet$. More precisely, for each p, n, we have a commutative diagram

$$\begin{tikzcd} F^pK^n \arrow[r, tail] \arrow[d, "d^n"'] & K^n \arrow[d, "d^n"] \\ F^pK^{n+1} \arrow[r, tail] & K^{n+1} \end{tikzcd}$$

and a chain

$$\cdots \longrightarrow F^{p+1}K^n \longrightarrow F^pK^n \longrightarrow F^{p-1}K^n \longrightarrow \cdots \longrightarrow K^n$$

Then they induces a chain on the cohomology objects

$$\cdots \longrightarrow H^n(F^{p+1}K^\bullet) \longrightarrow H^n(F^pK^\bullet) \longrightarrow H^n(F^{p-1}K^\bullet) \longrightarrow \cdots \longrightarrow H^n(K^\bullet)$$

The induced morphisms are not necessarily injective. Nevertheless, define

$$F^pH^n(K^\bullet) := \mathrm{Im}\,(H^n(F^pK^\bullet) \to H^n(K^\bullet))$$

which is a decreasing filtration of $H^n(K^\bullet)$. Our goal is to understand the $H^n(K^\bullet)$. However it usually cannot be attained. The second best is to understand its graded pieces, namely

$$\mathrm{Gr}^p_F(H^n(K)) := \frac{F^pH^n(K)}{F^{p+1}H^n(K)}$$

By definition, there is an exact sequence

$$0 \longrightarrow F^{p+1}H^n(K) \longrightarrow F^pH^n(K) \longrightarrow \mathrm{Gr}^p_F(H^n(K)) \longrightarrow 0$$

A general belief is that if we can understand the first and the third term, we can more or less capture the middle term. In the following, for convenience, put $K = K^\bullet$ and $F^p = F^pK^\bullet$. Consider a commutative triangle

$$\begin{tikzcd} H^n(F^{p+1}) \arrow[r] \arrow[dr] & H^n(K) \\ & H^n(F^p) \arrow[u] \end{tikzcd}$$

What we are interested in is the quotient of the vertical image by the horizontal image. For this, we use some algebras. Consider the short exact of complexes $0 \to F^{p+1} \to K \to K/F^{p+1} \to 0$. Then the induced sequence can extend the horizontal part of the triangle

$$\begin{tikzcd} H^n(F^{p+1}) \arrow[r] \arrow[dr] & H^n(K) \arrow[r] & H^n(K/F^{p+1}) \\ & H^n(F^p) \arrow[u] \arrow[ur] & \end{tikzcd}$$

Lemma 6.1. Given a commutative diagram with the top row being exact

$$\begin{tikzcd}
A'' \arrow[r, "\varphi''"] \arrow[dr] & A \arrow[r] & B \\
& A' \arrow[u, "\varphi'"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

the induced sequence

$$0 \longrightarrow \operatorname{Im}\varphi'' \longrightarrow \operatorname{Im}\varphi' \longrightarrow \operatorname{Im}\psi \longrightarrow 0$$

is exact.

Proof. By commutativity, we see the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\varphi'' \arrow[r] \arrow[d, hook] & \operatorname{Im}\varphi' \arrow[r] \arrow[dl, hook] & \operatorname{Im}\psi \arrow[r] & 0 \\
& A & & A' \arrow[ul, two heads] \arrow[u, two heads] &
\end{tikzcd}$$

This shows the exactness at $\operatorname{Im}\varphi''$ and $\operatorname{Im}\psi$. For the second place, consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\varphi'' \arrow[r] & A \arrow[r] & B & \\
& & \operatorname{Im}\varphi' \arrow[r] \arrow[u, hook] & \operatorname{Im}\psi \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Using functor of points, one can readily see that the kernel of the right-bottom map is $\operatorname{Im}\varphi''$, namely,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im}\varphi'' \arrow[r] & A \arrow[r] & B & \\
0 \arrow[r] & \operatorname{Im}\varphi'' \arrow[r] \arrow[u, equal] & \operatorname{Im}\varphi' \arrow[r] \arrow[u, hook] & \operatorname{Im}\psi \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

is commutative with exact rows. $\square$

Return to our discussion on the diagram

$$\begin{tikzcd}
H^n(F^{p+1}) \arrow[r] \arrow[dr] & H^n(K) \arrow[r] & H^n(K/F^{p+1}) \\
& H^n(F^p) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

By Lemma, $\operatorname{Gr}_F^p H^n(K) \cong \operatorname{Im}\left(H^n(F^p) \to H^n(K/F^{p+1})\right)$. On the other hand, using the short exact sequences

$$0 \to F^p/F^{p+1} \to K/F^{p+1} \to K/F^p \to 0$$

$$0 \to F^p \to K \to K/F^p \to 0$$

we can extend previous diagram to the following commutative diagram

$$
\begin{tikzcd}
H^n(F^{p+1}) \arrow[r] \arrow[dr] & H^n(K) \arrow[r] & H^n(K/F^{p+1}) \\
& H^n(F^p) \arrow[u] \arrow[ur] \arrow[r] & H^n(F^p/F^{p+1}) \arrow[u] \\
& H^{n-1}(K/F^p) \arrow[u, "*"'] \arrow[r, equal] & H^{n-1}(K/F^p) \arrow[u, "*"']
\end{tikzcd}
$$

with the bottom row, middle and the rightmost column being exact. $*$ means the morphisms are connecting homomorphisms. We draw it in a cuter way.

$$
\begin{tikzcd}
H^n(F^{p+1}) \arrow[r] \arrow[dr] & H^n(K) \arrow[r] & H^n(K/F^{p+1}) \\
& H^n(F^p) \arrow[u] \arrow[ur] \arrow[r] & H^n(F^p/F^{p+1}) \arrow[u] \\
& & H^{n-1}(K/F^p) \arrow[u, "*"'] \arrow[ul, "*"]
\end{tikzcd}
$$

Successive uses of Lemma give

$$
\operatorname{Gr}_F^p H^n(K) \cong \operatorname{Im}\left(H^n(F^p) \to H^n(K/F^{p+1})\right) \cong \frac{\operatorname{Im}(H^n(F^p) \to H^n(F^p/F^{p+1}))}{\operatorname{Im}(H^{n-1}(K/F^p) \xrightarrow{*} H^n(F^p/F^{p+1}))}
$$

Now, put

$$
\begin{aligned}
Z_\infty^{p,(n)} &:= \operatorname{Im}(H^n(F^p) \to H^n(F^p/F^{p+1})) \\
B_\infty^{p,(n)} &:= \operatorname{Im}(H^{n-1}(K/F^p) \xrightarrow{*} H^n(F^p/F^{p+1})) \\
E_\infty^{p,(n)} &:= Z_\infty^p \big/ B_\infty^p
\end{aligned}
$$

so what we obtain becomes $\operatorname{Gr}_F^p H^n(K) \cong E_\infty^{p,(n)} = Z_\infty^p \big/ B_\infty^p$. One may ask, suggested by our notation ∞, whether there exist any intermediate terms. The answer is positive, explained as follows. Look at the part

$$
\begin{tikzcd}
H^n(F^p) \arrow[r] & H^n(F^p/F^{p+1}) \\
& H^{n-1}(K/F^p) \arrow[u, "*"'] \arrow[ul, "*"]
\end{tikzcd}
$$

In fact, we have a very long triangle

$$
\begin{tikzcd}
H^n(F^p/F^{p+2}) \arrow[ddrrr] & & & & & \\
& H^n(F^p/F^{p+r}) \arrow[ul] \arrow[drr] & & & & \\
& & H^n(F^p) \arrow[ul] \arrow[r] & H^n(F^p/F^{p+1}) & & \\
& & & H^{n-1}(K/F^p) \arrow[ul, "*"] \arrow[u, "*"] & & \\
& & & & H^{n-1}(F^{p-r+1}/F^p) \arrow[ul] \arrow[uul, "*"] & \\
& & & & & H^{n-1}(F^2 \arrow[ul] \arrow[uuull, "*"]
\end{tikzcd}
$$

via the natural morphism $F^p/F^{p+r} \to F^p/F^{p+1}$ and the short exact sequence

$$0 \longrightarrow F^p/F^{p+1} \longrightarrow F^{p-r+1}/F^{p+1} \longrightarrow F^{p-r+1}/F^p \longrightarrow 0$$

With the diagram above, we define

$$Z_r^{p,(n)} := \mathrm{Im}(H^n(F^p/F^{p+r}) \to H^n(F^p/F^{p+1}))$$
$$B_r^{p,(n)} := \mathrm{Im}(H^{n-1}(F^{p-r+1}/F^p) \xrightarrow{*} H^n(F^p/F^{p+1}))$$

Hence we obtain a chain of sub-objects of $H(F^p/F^{p+1}) = H(\mathrm{Gr}_F^p K)$

$$Z_2^{p,(n)} \supseteq Z_3^{p,(n)} \supseteq \cdots \supseteq Z_r^{p,(n)} \supseteq \cdots \supseteq Z_\infty^{p,(n)} \supseteq B_\infty^{p,(n)} \supseteq \cdots \supseteq B_r^\infty \supseteq \cdots \supseteq B_3^{p,(n)} \supseteq B_2^{p,(n)}$$

This strikes a resemblance to the nested intervals. Finally, define

$$E_r^{p,(n)} = Z_r^{p,(n)}/B_r^{p,(n)}$$

Next, we discuss $Z_r^{p,(n)}$, $B_r^{p,(n)}$. Consider the diagram

$$
\begin{tikzcd}
H^n(F^p/F^{p+r+1}) \arrow[r] \arrow[dr] & H^n(F^p/F^{p+1}) \\
& H^n(F^p/F^{p+r}) \arrow[u]
\end{tikzcd}
$$

induced from the diagram without H. We mimic what we did before. Using the exact sequence $0 \to F^{p+1}/F^{p+r+1} \to F^p/F^{p+r+1} \to F^p/F^{p+1} \to 0$, we obtain

$$
\begin{tikzcd}
H^n(F^p/F^{p+r+1}) \arrow[r] \arrow[dr] & H^n(F^p/F^{p+1}) \arrow[r, "*"] & H^{n+1}(F^{p+1}/F^{p+r+1}) \\
& H^n(F^p/F^{p+r}) \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Using Lemma again, we obtain an isomorphism

$$\operatorname{Im}(H^n(F^p/F^{p+r}) \to H^{n+1}(F^{p+1}/F^{p+r+1})) \cong \frac{\operatorname{Im} H^n(F^p/F^{p+r} \to H^n(F^p/F^{p+1}))}{\operatorname{Im} H^n(F^p/F^{p+r} \to H^n(F^p/F^{p+r+1})/F^{p+1}))} = \frac{Z_r^{p,(n)}}{Z_{r+1}^{p,(n)}}$$

Also, from the commutative diagrams

$$\begin{tikzcd}
& 0 & & & \\
& F^{p+1}/F^{p+r} \arrow[u] & & F^p/F^{p+r} \arrow[ur] & \\
0 \arrow[r] & F^{p+1}/F^{p+r+1} \arrow[r] \arrow[u] & F^p/F^{p+r+1} \arrow[r] \arrow[ur] & F^p/F^{p+1} \arrow[r] & 0 \\
0 \arrow[r] & F^{p+r}/F^{p+r+1} \arrow[r] \arrow[u] \arrow[ur] & F^p/F^{p+r+1} \arrow[r] \arrow[u, equal] & F^p/F^{p+r} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & & &
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
H^n(F^p/F^{p+r+1}) \arrow[r] \arrow[dr] & H^n(F^p/F^{p+1}) \arrow[r, "*"] & H^{n+1}(F^{p+1}/F^{p+r+1}) \\
& H^n(F^p/F^{p+r}) \arrow[u] \arrow[ur] \arrow[r, "*"] & H^{n+1}(F^{p+r}/F^{p+r+1}) \arrow[u] \\
& & H^n(F^{p+1}/F^{p+r}) \arrow[u, "*"'] \arrow[ul]
\end{tikzcd}$$

Hence,

$$\operatorname{Im}(H^n(F^p/F^{p+r}) \to H^{n+1}(F^{p+1}/F^{p+r+1})) \cong \frac{\operatorname{Im}(H^n(F^p/F^{p+r}) \xrightarrow{*} H^{n+1}(F^{p+r}/F^{p+r+1}))}{\operatorname{Im}(H^n(F^{p+1}/F^{p+r}) \xrightarrow{*} H^{n+1}(F^{p+r}/F^{p+r+1}))} = \frac{B_{r+1}^{p+r,(n+1)}}{B_r^{p+r,(n+1)}}$$

In conclusion, there is a canonical isomorphism

$$\frac{Z_r^{p,(n)}}{Z_{r+1}^{p,(n)}} \cong \frac{B_{r+1}^{p+r,(n+1)}}{B_r^{p+r,(n+1)}}$$

Now, consider the composition

$$\begin{tikzcd}
\dfrac{Z_r^{p,(n)}}{B_r^{p,(n)}} \arrow[r, two heads] \arrow[d, equal] & \dfrac{Z_r^{p,(n)}}{Z_{r+1}^{p,(n)}} \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & \dfrac{B_{r+1}^{p+r,(n+1)}}{B_r^{p+r,(n+1)}} \arrow[r, tail] & \dfrac{Z_r^{p+r,(n+1)}}{B_r^{p+r,(n+1)}} \arrow[d, equal] \\
E_r^{p,(n)} \arrow[rrr, "d_r^{p,(n)}"'] & & & E_r^{p+r,(n+1)}
\end{tikzcd}$$

Note that $\operatorname{Im} d_r^{p,(n)} = Z_r^{p,(n)} / Z_{r+1}^{p,(n)} \cong B_{r+1}^{p+r,(n+1)} / B_r^{p+r,(n+1)}$. Observe that $d_r^{p,(n)} \circ d_r^{p-r,(n-1)} = 0$ from the diagram

$$\begin{tikzcd}
\frac{Z_r^{p-r,(n-1)}}{B_r^{p-r,(n-1)}} \arrow[r, two heads] \arrow[d, equal] & \frac{Z_r^{p-r,(n-1)}}{Z_{r+1}^{p-r,(n-1)}} \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & \frac{B_{r+1}^{p,(n)}}{B_r^{p,(n)}} \arrow[r, tail] & \frac{Z_r^{p,(n)}}{B_r^{p,(n)}} \arrow[r, two heads] \arrow[d, equal] & \frac{Z_r^{p,(n)}}{Z_{r+1}^{p,(n)}} \arrow[r, "\sim"] \arrow[d, phantom, "\circlearrowleft"] & \frac{B_{r+1}^{p+r,(n+1)}}{B_r^{p+r,(n+1)}} \arrow[r, tail] & \frac{Z_r^{p+r,(n+1)}}{B_r^{p+r,(n+1)}} \arrow[d, equal] \\
E_r^{p-r,(n-1)} \arrow[rrr, "d_r^{p-r,(n-1)}"'] & & & E_r^{p,(n)} \arrow[rrr, "d_r^{p,(n)}"'] & & & E_r^{p+r,(n+1)}
\end{tikzcd}$$

so (E,d) forms a complex. Notice that $\ker d_r^{p,(n)} = Z_{r+1}^{p,(n)}/B_r^{p,(n)}$ and $\operatorname{Im} d_r^{p-r,(n-1)} = B_{r+1}^{p,(n)}/B_r^{p,(n)}$. Hence we obtain

$$\frac{\ker d_r^{p,(n)}}{\operatorname{Im} d_r^{p-r,(n-1)}} = \frac{Z_{r+1}^{p,(n)}/B_r^{p,(n)}}{B_{r+1}^{p,(n)}/B_r^{p,(n)}} \cong \frac{Z_{r+1}^{p,(n)}}{B_{r+1}^{p,(n)}} = E_{r+1}^{p,(n)}$$

This special structure of $(E_r^{p,(n)}, d_r^{p,(n)})$ suggests us to view r as the "page number", and view taking cohomology of $(E_r^{p,(n)}, d_r^{p,(n)})$ at p-th position as "turning to the next page and locating the same place".

Now we make a slight change on the notation. Put $q = n-p$, and change $(p,(n))$ to (p,q). Namely, we now write $Z_r^{p,q}$, $B_r^{p,q}$, $E_r^{p,q}$, $d_r^{p,q}$, $H^{p+q}(F^p)$. We call $(E_r^{p,q}, d_r^{p,q})$ with the isomorphisms $\dfrac{\ker d_r^{p,q}}{\operatorname{Im} d_r^{p-r,q+r-1}} \cong E_{r+1}^{p,q}$ a **spectral sequence**.

- We write $E_2^{p,q} \Rightarrow_p H^{p+q}(K^\bullet)$ for the isomorphism

$$E_\infty^{p,q} \cong \operatorname{Gr}_F^p(H^n(K)) := \frac{F^pH^n(K)}{F^{p+1}H^n(K)}$$

  for each p,q, and say that the spectral sequence (E,d) **converges** to the filtered objected $H^n(K)$.

We return to the discussion on the graded pieces. By definition,

$$\operatorname{Gr}_F^p H^{p+q}(K^\bullet) = \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = \frac{\operatorname{Im}(H^{p+q}(F^p) \to H^{p+q}(F^p/F^{p+1}))}{\operatorname{Im}(H^{p+q-1}(K/F^p) \xrightarrow{*} H^{p+q}(F^p/F^{p+1}))}$$

and

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} = \frac{\operatorname{Im}(H^{p+q}(F^p/F^{p+r}) \to H^{p+q}(F^p/F^{p+1}))}{\operatorname{Im}(H^{p+q-1}(F^{p-r+1}/F^p) \xrightarrow{*} H^{p+q}(F^p/F^{p+1}))}$$

Domains of their numerators can be obtained from the following diagrams.

$$\begin{tikzcd}
\cdots \arrow[r] & F^{p+r}K^{p+q-1} \arrow[r] \arrow[d] & F^{p+r}K^{p+q} \arrow[r] \arrow[d] & F^{p+r}K^{p+q+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^pK^{p+q-1} \arrow[r] \arrow[d] & F^pK^{p+q} \arrow[r] \arrow[d] & F^pK^{p+q+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & F^pK^{p+q-1}/F^{p+r}K^{p+q-1} \arrow[r] & F^pK^{p+q}/F^{p+r}K^{p+q} \arrow[r] & F^pK^{p+q+1}/F^{p+r}K^{p+q+1} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose for each $n \in \mathbb{Z}$, there exists $p_0 = p_0(n)$ such that $F^pK^n = 0$ for all $p \geqslant p_0$. Then when $r \gg 1$, we have $F^{p+r}K^{p+q} = 0$; hence $Z_r^{p,q} = Z_\infty^{p,q}$.

Imagine another diagram that involves domain of the denominators. Suppose also for each $n \in \mathbb{Z}$, there exists $p_1 = p_1(n)$ such that $F^pK^n = K^n$ for all $p \leqslant p_1$. Then when $r \gg 1$, we have $F^{p-r+1}K^{p+q} = K^{p+q}$; hence $B_r^{p,q} = B_\infty^{p,q}$.

Let us combine these two situation. Then for each $n \in \mathbb{N}$, the filtration F^pK^n is finite, in the sense that there exists integers p_0, p_1, depending on n such that

$$K^n = F^{p_1}K^n \supseteq F^{p_1+1}K^n \supseteq \cdots \supseteq F^{p_0-1}K^n \supseteq F^{p_0}K^n = 0$$

Then when $r \gg 1$, $Z_r^{p,q} = Z_\infty^{p,q}$ and $B_r^{p,q} = B_\infty^{p,q}$, and hence $E_r^{p,q} = \operatorname{Gr}_F^p H^{p+q}(K^\bullet)$.

This condition seems to be very strong, but in fact in many practical examples arising from algebraic geometry and algebraic topology, the filtrations do have this property.

Example. Consider the **double complex** (K, d', d'')

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & K^{p,q+1} \arrow[r, "d'"] \arrow[u] & K^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & K^{p,q} \arrow[r, "d'"'] \arrow[u, "d''"] \arrow[ur, phantom, "\circlearrowleft"] & K^{p+1,q} \arrow[r] \arrow[u, "d''"'] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

that is, $(K^{p,\bullet}, d'')$ and $(K^{\bullet,q}, d')$ are complexes with each square commutative. We form its **total complex** $\operatorname{tot} K$ as follows.

- For each $n \in \mathbb{Z}$, $(\operatorname{tot} K)^n := \prod_{i+j=n} K^{i,j}$
- For each $n \in \mathbb{Z}$, the differential $d : (\operatorname{tot} K)^n \to (\operatorname{tot} K)^{n+1}$ is given by

$$d = \sum_{i+j=n} (-1)^i d'' + d'$$

  Here i shall be viewed as the p-coordinate.

A decreasing filtration of $\operatorname{tot}(K)$ is given by the subcomplexes $F^p(\operatorname{tot} K)^\bullet$ ($p \in \mathbb{Z}$), defined by

$$F^p(\operatorname{tot} K)^n := \prod_{\substack{i+j=n \\ i \geqslant p}} K^{i,j}$$

This is obtained by dropping those $K^{i,j}$ with $i < p$ and collecting those $i \geqslant p$. From this filtration, we can construct, as above, a spectral sequence $(E_r^{p,q}, d_r^{p,q})$ with

$$E_\infty^{p,q} \cong \operatorname{Gr}_F^p H^{p+q}(\operatorname{tot}(K)) = \frac{F^p H^{p+q} \operatorname{tot}(K)}{F^{p+1} H^{p+q} \operatorname{tot}(K)}$$

When K is a **first quadrant double complex**, that is, $K^{p,q} = 0$ when $p < 0$ or $q < 0$, then

$$(\operatorname{tot} K)^n = \prod_{i+j=n} K^{i,j} = \bigoplus_{i+j=n} K^{i,j}$$

and $F^p(\operatorname{tot} K)^n$ is a finite filtration of $\operatorname{tot} K$. Generally,

Proposition 6.2. Suppose that for all $m \in \mathbb{Z}$, $K^{i,m-i} = 0$ for all but finitely many i. Then

1. For each n, $F^p \operatorname{tot}(K)^n = 0$ and $F^{p'} \operatorname{tot}(K)^n = \operatorname{tot}(K)^n$ for some p, p'.

2. For each p, q, we have $Z_r^{p,q} = Z_\infty^{p,q}$ and $B_r^{p,q} = B_\infty^{p,q}$ for some r.

In particular, for each $p, q \in \mathbb{Z}$, if $r \gg 2$,

$$E_r^{p,q} \cong E_\infty^{p,q} = \operatorname{Gr}_F^p H^{p+q}(\operatorname{tot}(K)^\bullet)$$

Proof. By assumption, for each n, we can find two numbers $s = s(n) < t = t(n)$ such that $K^{i,n-i} = 0$ for $i < s$ or $i > t$. Then for $p' < s$, $F^{p'} \operatorname{tot}(K)^n = \operatorname{tot}(K)^n$, and for $p > t$, we have $F^p \operatorname{tot}(K)^n = 0$.

In computing the following images

$$Z_r^{p,q} := \operatorname{Im}(H^{p+q}(F^p/F^{p+r}) \to H^{p+q}(F^p/F^{p+1}))$$
$$B_r^{p,q} := \operatorname{Im}(H^{p+q-1}(F^{p-r+1}/F^p) \to H^{p+q}(F^p/F^{p+1}))$$

we see

- if $p + r > t(p+q)$, then $F^{p+r} \operatorname{tot}(K)^{p+q} = 0$, so $Z_r^{p,q} = Z_\infty^{p,q}$ in this case.

- if $p - r + 1 < s(p+q-1)$, then $F^{p-r+1} \operatorname{tot}(K)^{p+q-1} = \operatorname{tot}(K)^{p+q-1}$, so $B_r^{p,q} = B_\infty^{p,q}$.

When r is large enough, both conditions can be satisfied, and the result follows. □

Suppose (E, d) is a spectral sequence with $E_2^{p,q} \Rightarrow_p H^{p+q}$, where H is filtered by $F^\bullet H$. We say (E, d) is **biregular** if

1. for each n, there exists p, p' such that $F^p H^n = H^n$ and $F^{p'} H^n = 0$;

2. for each p, q, we have $Z_r^{p,q} = Z_\infty^{p,q}$ and $B_r^{p,q} = B_\infty^{p,q}$ for some r.

Corollary 6.2.1. Under the assumption of Proposition, there exists a biregular spectral sequence (E, d) converging to $H(\operatorname{tot} K)$.

6.1 the E_2 page

By the very definition, the numerator of the quotient

$$E_2^{p,q} = \frac{Z_2^{p,q}}{B_2^{p,q}} = \frac{\operatorname{Im}(H^{p+q}(F^p/F^{p+2}) \to H^{p+q}(F^p/F^{p+1}))}{\operatorname{Im}(H^{p+q-1}(F^{p-1}/F^p) \xrightarrow{*} H^{p+q}(F^p/F^{p+1}))}$$

comes from the long exact sequence induced by the short exact sequence

$$0 \longrightarrow F^{p+1}/F^{p+2} \longrightarrow F^p/F^{p+2} \longrightarrow F^p/F^{p+1} \longrightarrow 0$$

so

$$Z_2^{p,q} \cong \ker(H^{p+q}(F^p/F^{p+1}) \xrightarrow{*} H^{p+q+1}(F^{p+1}/F^{p+2}))$$

Consider the complex form by the connecting homomorphisms

$$\cdots \xrightarrow{\;*\;} H^{p+q-1}(F^{p-1}/F^p) \xrightarrow{\;*\;} H^{p+q}(F^p/F^{p+1}) \xrightarrow{\;*\;} H^{p+q+1}(F^{p+1}/F^{p+2}) \xrightarrow{\;*\;} \cdots$$

Then what we do above implies

$$E_2^{p,q} \cong H^{p+q}(H^\bullet(\operatorname{Gr}_F^{\bullet-q}(K)), *)$$

Let us specialize to double complexes. In this situation, the graded pieces are fairly simple: we have the following exact sequence of complexes

$$0 \to F^{p+1}\operatorname{tot}(K)^\bullet \to F^p\operatorname{tot}(K)^\bullet \to (K^{p,\bullet-p}, (-1)^p d'') \to 0$$

The rightmost term is just original double complex, but forgetting the horizontal differentials. Hence

$$H^{p+q}(\operatorname{Gr}_F^p\operatorname{tot}(K)^\bullet) \cong H^{p+q}(K^{p,\bullet-p}, (-1)^p d'') = H^q(K^{p,\bullet}, d'')$$

Now we are in the situation

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\;*\;} & H^{p+q-1}(\operatorname{Gr}_F^{p-1}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & H^{p+q}(\operatorname{Gr}_F^{p}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & H^{p+q+1}(\operatorname{Gr}_F^{p+1}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & \cdots \\
 & & \| & & \| & & \| & & \\
 & & H^q(K^{p-1,\bullet}, d'') & & H^q(K^{p,\bullet}, d'') & & H^q(K^{p+1,\bullet}, d'') & &
\end{array}$$

On the other hand, since d' can be view as chain maps between $(K^{p,\bullet}, d')$, d' induces the maps on cohomology objects

$$\cdots \longrightarrow H^q(K^{p-1,\bullet}, d'') \xrightarrow{\;d'\;} H^q(K^{p,\bullet}, d'') \xrightarrow{\;d'\;} H^q(K^{p+1,\bullet}, d'') \longrightarrow \cdots$$

It is natural to ask whether or not this complex can be fitted into the above complex of graded pieces. The answer is positive, and it can be proved by the definition of the connecting homomorphism.

Proposition 6.3. We have the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\;*\;} & H^{p+q-1}(\operatorname{Gr}_F^{p-1}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & H^{p+q}(\operatorname{Gr}_F^{p}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & H^{p+q+1}(\operatorname{Gr}_F^{p+1}\operatorname{tot}(K)^\bullet) & \xrightarrow{\;*\;} & \cdots \\
 & & \| & & \| & & \| & & \\
\cdots & \longrightarrow & H^q(K^{p-1,\bullet}, d'') & \xrightarrow{\;d'\;} & H^q(K^{p,\bullet}, d'') & \xrightarrow{\;d'\;} & H^q(K^{p+1,\bullet}, d'') & \longrightarrow & \cdots
\end{array}$$

Proof. Let us focus on one block.

$$\begin{tikzcd} H^{p+q}(\mathrm{Gr}_F^p \operatorname{tot}(K)^\bullet) \arrow[r, "*"] \arrow[d, equal] & H^{p+q+1}(\mathrm{Gr}_F^{p+1} \operatorname{tot}(K)^\bullet) \arrow[d, equal] \\ H^q(K^{p,\bullet}, d'') \arrow[r, "d'"] & H^q(K^{p+1,\bullet}, d'') \end{tikzcd}$$

The upper horizontal map comes from the short exact sequence of complexes

$$0 \longrightarrow F^{p+1}/F^{p+2} \longrightarrow F^p/F^{p+2} \longrightarrow F^p/F^{p+1} \longrightarrow 0$$

Take $x \in F^p \operatorname{tot}(K)^n / F^{p+1} \operatorname{tot}(K)^n$ with $d''(x) = 0$. Now consider $x \in F^p \operatorname{tot}(K)^n / F^{p+2} \operatorname{tot}(K)^n$, and send it to $F^p \operatorname{tot}(K)^{n+1} / F^{p+2} \operatorname{tot}(K)^{n+1}$; the result is $d'(x)$. Since $d'(x) \in F^{p+1} \operatorname{tot}(K)^{n+1} / F^{p+2} \operatorname{tot}(K)^{n+1}$, we finally find that

$$\left(H^{p+q}(\mathrm{Gr}_F^p \operatorname{tot}(K)^\bullet) \xrightarrow{*} H^{p+q+1}(\mathrm{Gr}_F^{p+1} \operatorname{tot}(K)^\bullet)\right)([x]) = [d'(x)]$$

proving that commutativity. □

Corollary 6.3.1. $E_2^{p,q} \cong H^p(H^q(K^{\bullet,\bullet}, d''), d')$; more beautifully

$$E_2^{p,q} \cong H^p_{d'}(H^q_{d''}(K))$$

6.2 Applications I

Example (Augmented double complexes)**.** Consider the following double complex

$$\begin{tikzcd} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 \arrow[r] & F^3 \arrow[r] \arrow[u] & K^{0,3} \arrow[r] \arrow[u] & K^{1,3} \arrow[r] \arrow[u] & K^{2,3} \arrow[r] \arrow[u] & K^{3,3} \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & F^2 \arrow[r] \arrow[u] & K^{0,2} \arrow[r] \arrow[u] & K^{1,2} \arrow[r] \arrow[u] & K^{2,2} \arrow[r] \arrow[u] & K^{3,2} \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & F^1 \arrow[r] \arrow[u] & K^{0,1} \arrow[r] \arrow[u] & K^{1,1} \arrow[r] \arrow[u] & K^{2,1} \arrow[r] \arrow[u] & K^{3,1} \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & F^0 \arrow[r] \arrow[u] & K^{0,0} \arrow[r] \arrow[u] & K^{1,0} \arrow[r] \arrow[u] & K^{2,0} \arrow[r] \arrow[u] & K^{3,0} \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

with all rows exact. We have a natural morphism $F \to \operatorname{tot}(K)$. We claim this is a **quasi-isomorphism**, that is,

$$H^\bullet(F) \to H^\bullet(\operatorname{tot}(K))$$

is an isomorphism. We split it into two first-quadrant double complexes, namely,

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
F^3 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
F^2 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
F^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
F^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
\qquad
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K^{0,3} & \longrightarrow & K^{1,3} & \longrightarrow & K^{2,3} & \longrightarrow & K^{3,3} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K^{0,2} & \longrightarrow & K^{1,2} & \longrightarrow & K^{2,2} & \longrightarrow & K^{3,2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K^{0,1} & \longrightarrow & K^{1,1} & \longrightarrow & K^{2,1} & \longrightarrow & K^{3,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
K^{0,0} & \longrightarrow & K^{1,0} & \longrightarrow & K^{2,0} & \longrightarrow & K^{3,0} & \longrightarrow & \cdots
\end{array}$$

and the original morphisms connecting F and K are extended to a morphism between the two double complexes. The induced morphism on total complexes coincides with $F \to \mathrm{tot}(K)$, and it induces morphisms between the induced spectral sequences. (Note that the construction of a spectral sequence is functorial!)

By transposing the double complexes above, we compute their E_2 pages by first computing the horizontal cohomology and then compute the vertical cohomology of resulting complexes (this simply means we use a different filtration; explicitly we use

$$F^p \,\mathrm{tot}(K)^n = \prod_{\substack{i+j=n \\ j \geqslant p}} K^{i,j}$$

to filter the total complex). By our assumption on exactness, we see the above complexes have the same horizontal cohomology, all being the left one. Hence they have the same E_2 page:

$$E_2^{p,q} = \begin{cases} H^q(F) & , p \geqslant 0,\, q = 0 \\ 0 & , \text{else} \end{cases}$$

This also means the filtrations on $\mathrm{tot}(F) \cong F$ and $\mathrm{tot}(K)$ have only one jump, i.e.,

$$H^n(F) = F^0 H^n(F) \supseteq F^1 H^n(F) = 0$$

$$H^n(\mathrm{tot}(K)) = F^0 H^n(\mathrm{tot}(K)) \supseteq F^1 H^n(\mathrm{tot}(K)) = 0$$

and that $E_2^{p,q} = E_\infty^{p,q}$ for all p, q. Hence

$$\begin{tikzcd}
H^n(F) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & H^n(\mathrm{tot}(K)) \arrow[d, equal] \\
F^0H^n(F) \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & F^0H^n(\mathrm{tot}(K)) \arrow[d, equal] \\
E_\infty^{0,n} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & E_\infty^{0,n} \arrow[d, equal] \\
E_2^{0,n} \arrow[r, "\sim"'] & E_n^{0,n}
\end{tikzcd}$$

Example (Tor functor)**.** Let R be a ring and M, N be R-modules. We show that $\mathrm{Tor}_n^R(M,N) = \mathrm{Tor}_n^R(N,M)$. To compute the Tor, pick

$$P : \cdots \to P_2 \to P_1 \to P_0 \to 0$$

$$Q : \cdots \to Q_2 \to Q_1 \to Q_0 \to 0$$

to be deleted free resolutions of M and N, respectively. Consider the tensor product $P \otimes Q$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \otimes Q_0 \arrow[r] & P_1 \otimes Q_0 \arrow[r] & P_0 \otimes Q_0 \\
\cdots \arrow[r] & P_2 \otimes Q_1 \arrow[r] \arrow[u] & P_1 \otimes Q_1 \arrow[r] \arrow[u] & P_0 \otimes Q_1 \arrow[u] \\
\cdots \arrow[r] & P_2 \otimes Q_2 \arrow[r] \arrow[u] & P_1 \otimes Q_2 \arrow[r] \arrow[u] & P_0 \otimes Q_2 \arrow[u] \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

We compute the E^2 page of the induced spectral sequence. The vertical homology is

$$H_q(P_p \otimes Q_\bullet) = P_p \otimes H_q(Q_\bullet) = \begin{cases} P_p \otimes N & , q = 0 \\ 0 & , q \neq 0 \end{cases}$$

and the horizontal homology of the resulted complex is

$$E_{p,q}^2 = H_pH_q(P_\bullet \otimes Q_\bullet) = \begin{cases} H_p(P_\bullet \otimes N) & , q = 0 \\ 0 & , q \neq 0 \end{cases}$$

This then implies that $E_{p,q}^2 = E_{p,q}^\infty$, and the filtration $H^{p+q}(\mathrm{tot}(P \otimes Q))$ has only one nonzero piece (only one jump). Hence

$$H_p(P_\bullet \otimes N) = E_2^{p,0} = E_\infty^{p,0} = H_p(\mathrm{tot}(P \otimes Q))$$

By transposing $P \otimes Q$, we can obtain

$$H_p(M \otimes Q_\bullet) = H_p(\operatorname{tot}(P \otimes Q))$$

Hence we obtain the desired isomorphism

$$H_p(P \otimes N) = H_p(\operatorname{tot}(P \otimes Q)) = H_p(M \otimes Q)$$

6.3 Associated five-term exact sequences

Proposition 6.4. Suppose that we have a biregular spectral sequence (E, d) that converges to H filtered by FH, namely, for each $p, q \in \mathbb{N}$ there exists r with

$$E_r^{p,q} = E_\infty^{p,q} \cong \operatorname{Gr}_F^p H^{p+q} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

Moreover, suppose there exists $n \in \mathbb{N}$ such that $E_2^{p,q} = 0$ if either (i) $p < 0$, (ii) $q < 0$, or (iii) $0 < q < n$. Then

1. $E_2^{p,0} \cong H^p$ for $p < n$.

2. there exists a functorial exact sequence

$$0 \longrightarrow E_2^{n,0} \longrightarrow H^n \longrightarrow E_2^{0,n} \longrightarrow E_2^{n+1,0} \longrightarrow H^{n+1}$$

To see 1., we know the lines with slope $\dfrac{1-r}{r}$ $(r \geqslant 2)$ passing through $(p, 0)$, $p < 0$ contain only trivial $E_r^{p,q}$ (except for $E_r^{p,0}$). Hence $E_2^{p,0} = \cdots = E_r^{p,0} = E_\infty^{p,0}$, and also $E_2^{p-t,t} = \cdots = E_r^{p-t,t} = E_\infty^{p-t,t} = 0$ for $t \neq 0$. Hence

$$H^p = \cdots = F^{p+1} H^p = F^p H^p \supseteq F^{p-1} H^p = 0 = \cdots$$

so that

$$H^p = E_\infty^{p,0} = E_2^{p,0}$$

Now consider the case $p = n$, and look at $E_r^{n-t,t}$; the only nontrivial terms occur at $E_r^{0,n}$ and $E_r^{n,0}$. We have

$$E_2^{n-t,t} = \cdots = E_\infty^{n-t,t} = 0, \; t \neq 0, n$$

$$E_2^{n,0} = \cdots = E_\infty^{n,0}$$

$$E_2^{0,n} = E_3^{0,n} = \cdots = E_n^{0,n} = E_{n+1}^{0,n}$$

$$E_{n+2}^{0,n} = \ker(E_{n+1}^{0,n} \to E_{n+1}^{n+1,0})$$

$$E_{n+2}^{0,n} = E_{n+3}^{0,n} = \cdots = E_\infty^{0,n}$$

This turns out giving the information about the filtration:

$$H^n = \cdots = F^0 H^n \supseteq F^1 H^n = \cdots = F^{n-1} H^n = F^n H^n \supseteq F^{n+1} H^n = \cdots = 0$$

The case $p = n + 1$ gives

$$H^{n+1} = \cdots = F^0H^{n+1} \supseteq F^1H^{n+1} \supseteq F^2H^{n+1} = \cdots = F^{n+1}H^{n+1} \supseteq F^{n+2}H^{n+1} = \cdots = 0$$

and hence we have a sequence of maps

$$\begin{tikzcd}
0 \arrow[r] & F^1H \arrow[r] & H^n = F^0H^n \arrow[r, two heads] \arrow[drr] & E_\infty^{0,n} \arrow[r, equal] & E_{n+2}^{0,n} \arrow[d, tail] & & & \\
& & & & E_{n+1}^{0,n} \arrow[r] & E_{n+1}^{n+1,0} \arrow[r, two heads] & E_{n+2}^{n+1,0} \arrow[d, equal] & \\
& & & & & & E_\infty^{n+1,0} \arrow[d, equal] & \\
& & & & & & F^{n+1}H^{n+1} \arrow[r, tail] & F^0H^{n+1} \arrow[d, equal] \\
& & & & & & & H^{n+1}
\end{tikzcd}$$

Therefore, we have an exact sequence

$$0 \longrightarrow F^1H^n \longrightarrow H^n \longrightarrow E_{n+1}^{0,n} \longrightarrow E_{n+1}^{n+1,0} \longrightarrow H^{n+1}$$

Plugging the values $F^1H^n = E_\infty^{n,0} = E_2^{n,0}$, $E_{n+1}^{0,n} = E_\infty^{0,n}$ and $E_{n+1}^{n+1,0} = E_\infty^{n+1,0}$ give the desired exact sequence. This complete the proof.

Specializing to the case $n = 1$, we obtain

Corollary 6.4.1. Suppose we have a spectral sequence (E, d) converging to H filtered by FH such that

(i) for each n, there exists p, q with $F^pH^n = H^n$ and $F^qH^n = 0$;

(ii) $E_2^{p,q} = 0$ if $p < 0$ or $q < 0$.

Then we have a functorial exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2$$

6.4 Grothendieck spectral sequence

Proposition 6.5. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be abelian categories with $\mathcal{C}$, $\mathcal{C}'$ having enough injective objects, and

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$$

be two left exact (covariant) functors. Suppose for all injective objects $I \in \mathcal{C}$, $F(I)$ is G-acyclic, namely $RG^n(F(I)) = 0$ for $n > 0$. Then for all $C \in \mathcal{C}$, there exists a functorial biregular spectral sequence (E, d) such that

$$E_2^{p,q} = R^pG(R^qF(C)) \Rightarrow R^{p+q}(GF)(C)$$

To compute the right derived functor of F, first we pick an injective resolution of C, namely an exact sequence in $\mathcal{C}$

$$0 \longrightarrow C \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

with each I^n injective. Applying F to $0 \to I^0 \to I^1 \to \cdots$ we obtain a complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow \cdots$$

Next, to compute $GF(I^n)$, we must pick an injective resolution of each $F(I^n)$. We contend that there exists a double complex J with each term injective and

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
J^{0,2} \arrow[r] \arrow[u] & J^{1,2} \arrow[r] \arrow[u] & J^{2,2} \arrow[r] \arrow[u] & \cdots \\
J^{0,1} \arrow[r] \arrow[u] & J^{1,1} \arrow[r] \arrow[u] & J^{2,1} \arrow[r] \arrow[u] & \cdots \\
J^{0,0} \arrow[r] \arrow[u] & J^{1,0} \arrow[r] \arrow[u] & J^{2,0} \arrow[r] \arrow[u] & \cdots \\
F(I^0) \arrow[r] \arrow[u] & F(I^1) \arrow[r] \arrow[u] & F(I^2) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with each column exact. In fact, we can do better.

Lemma 6.6 (Cartan-Eilenberg injective resolution)**.** Suppose we have a complex $\mathbf{C} : C^0 \to C^1 \to C^2 \to \cdots$ in $\mathcal{C}$. We split it into many short exact sequences

$$0 \to Z^n(\mathbf{C}) \to C^n \to B^{n+1}(\mathbf{C}) \to 0$$

$$0 \to B^n(\mathbf{C}) \to Z^n(\mathbf{C}) \to H^n(\mathbf{C}) \to 0$$

Then there exists an double complex J such that

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
J^{0,2} \arrow[r] \arrow[u] & J^{1,2} \arrow[r] \arrow[u] & J^{2,2} \arrow[r] \arrow[u] & \cdots \\
J^{0,1} \arrow[r] \arrow[u] & J^{1,1} \arrow[r] \arrow[u] & J^{2,1} \arrow[r] \arrow[u] & \cdots \\
J^{0,0} \arrow[r] \arrow[u] & J^{1,0} \arrow[r] \arrow[u] & J^{2,0} \arrow[r] \arrow[u] & \cdots \\
C^{0} \arrow[r] \arrow[u] & C^{1} \arrow[r] \arrow[u] & C^{2} \arrow[r] \arrow[u] & \cdots \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with each column an injective resolution of the bottom object. Moreover, the induced complexes

$$0 \to Z^n(\mathbf{C}) \to Z^n(J^{\bullet,0}) \to Z^n(J^{\bullet,1}) \to Z^n(J^{\bullet,2}) \to \cdots$$

$$0 \to B^n(\mathbf{C}) \to B^n(J^{\bullet,0}) \to B^n(J^{\bullet,1}) \to B^n(J^{\bullet,2}) \to \cdots$$

$$0 \to H^n(\mathbf{C}) \to H^n(J^{\bullet,0}) \to H^n(J^{\bullet,1}) \to H^n(J^{\bullet,2}) \to \cdots$$

are also injective resolutions.

Proof. Let us start with

$$0 \to Z^0 \to C^0 \to B^1 \to 0$$

Pick injective resolutions of Z^0 and B^1; then we can simultaneously resolve C^0 injectively, namely, we are in the situation(left)

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & Z^{0,2} \arrow[r] \arrow[u] & J^{0,2} \arrow[r] \arrow[u] & B^{1,2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z^{0,1} \arrow[r] \arrow[u] & J^{0,1} \arrow[r] \arrow[u] & B^{1,1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z^{0,0} \arrow[r] \arrow[u] & J^{0,0} \arrow[r] \arrow[u] & B^{1,0} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & Z^{0} \arrow[r] \arrow[u] & C^{0} \arrow[r] \arrow[u] & B^{1} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
0 \arrow[r] & B^{1,2} \arrow[r] \arrow[u] & Z^{1,2} \arrow[r] \arrow[u] & H^{1,2} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B^{1,1} \arrow[r] \arrow[u] & Z^{1,1} \arrow[r] \arrow[u] & H^{1,1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B^{1,0} \arrow[r] \arrow[u] & Z^{1,0} \arrow[r] \arrow[u] & H^{1,0} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B^{1} \arrow[r] \arrow[u] & Z^{1} \arrow[r] \arrow[u] & H^{1} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with each row exact and each column an injective resolution. Next take a simultaneous injective resolution of $0 \to B^1 \to Z^1 \to H^1 \to 0$ (right). The result follows by repetition of the procedure. $\square$

Return to the proof. Now take J as in Lemma and apply G to the whole complex.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
GJ^{0,2} & \longrightarrow & GJ^{1,2} & \longrightarrow & GJ^{2,2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
GJ^{0,1} & \longrightarrow & GJ^{1,1} & \longrightarrow & GJ^{2,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
GJ^{0,0} & \longrightarrow & GJ^{1,0} & \longrightarrow & GJ^{2,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
G(F(I^0)) & \longrightarrow & G(F(I^1)) & \longrightarrow & G(F(I^2)) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Since $F(I^n)$ is G-acyclic and G is left exact, each column above is exact. By the first application in 1.2, we have an isomorphism $R^n(GF)(C) = H^n(GF)(I) \xrightarrow{\sim} H^n(\text{tot}(GJ))$. As always, we have a biregular spectral sequence converging to $H^n(\text{tot}(GJ))$, with

$$E_2^{p,q} \cong H^p H^q(GJ)$$

We contend that $H^q(GJ) \cong GH^q(J)$. Consider

$$0 \to Z^{p,q} \to J^{p,q} \to B^{p+1,q} \to 0$$

Since Z is injective,

$$0 \to GZ^{p,q} \to GJ^{p,q} \to GB^{p+1,q} \to 0$$

is exact, and thus

$$0 \to GB^{p,q} \to GZ^{p,q} \to H^q(GJ^{p,\bullet}) \to 0$$

is exact. On the other hand, we have

$$0 \to B^{p,q} \to Z^{p,q} \to H^q(J^{p,\bullet}) \to 0$$

Since B is exact, the complex

$$0 \to GB^{p,q} \to GZ^{p,q} \to GH^q(J^{p,\bullet}) \to 0$$

is exact. This demonstrates the contention $GH^q(J^{p,\bullet}) \cong H^q(GJ^{p,\bullet})$. Hence

$$E_2^{p,q} \cong H^p H^q(GJ) \cong H^p G(H^q J) \cong (R^p G)(H^q(F(I))) \cong (R^p G)(R^q F)(C)$$

Corollary 6.6.1. Under the same assumption as Theorem, we have a five-term exact sequence

$$0 \to R^1G(FC) \to R^1(GF)(C) \to G(R^1F(C)) \to R^2G(FC) \to R^2(GF)(C)$$

There are variants of Grothendieck spectral sequences.

Proposition 6.7. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be abelian categories with $\mathcal{C}$, $\mathcal{C}'$ having enough projective, and

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$$

be right exact functors such that for all projective objects $P \in \mathcal{C}$, $F(P)$ is G-acyclic. Then for all $C \in \mathcal{C}$, there exists a functorial biregular spectral sequence (E, d) such that

$$E^2_{p,q} = L_pG(L_qF(C)) \Rightarrow L_{p+q}(GF)(C)$$

Proposition 6.8. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be abelian categories with $\mathcal{C}$, $\mathcal{C}'$ where $\mathcal{C}$ has enough projective and $\mathcal{C}'$ has enough injective. Let

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$$

be such that G is contravariant left exact, F is covariant right exact and for all projective objects $P \in \mathcal{C}$, $F(P)$ is G-acyclic. Then for all $C \in \mathcal{C}$, there exists a functorial biregular spectral sequence (E, d) such that

$$E_2^{p,q} = R^pG(L_qF(C)) \Rightarrow R^{p+q}(GF)(C)$$

Proposition 6.9. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be abelian categories with $\mathcal{C}$, $\mathcal{C}'$ where $\mathcal{C}$ has enough injective and $\mathcal{C}'$ has enough projective. Let

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$$

be contravariant left exact functors such that for all projective objects $P \in \mathcal{C}$, $F(P)$ is G-acyclic. Then for all $C \in \mathcal{C}$, there exists a functorial biregular spectral sequence (E, d) such that

$$E^2_{p,q} = R^pG(R^qF(C)) \Rightarrow L_{p+q}(GF)(C)$$

6.5 Applications II

Example. Let R and S be rings (not necessarily commutative). For an abelian group M which has a left R-module structure, we write ${}_RM$; if it has a right R-module structure, we write M_R. We also write ${}_RM_S$ if M is an (R, S)-bimodule.

Now suppose we have three modules A_R, ${}_RB_S$ and ${}_SC$. Consider the functors

$$F = B \otimes_S - : {}_S\text{Mod} \to {}_R\text{Mod}$$
$$G = A \otimes_R - : {}_R\text{Mod} \to \text{Ab}$$

Then F and G are right exact, and by the associativity

$$A \otimes_R (B \otimes_S C) \cong (A \otimes_R B) \otimes_S C$$

we can view $GF = (A \otimes_R B) \otimes_S -$. To obtain a Grothendieck spectral sequence, assume that for every projective ${}_SP$, $\mathrm{Tor}_i^R(A, B \otimes_S P) = 0$ for all $i \geqslant 1$. Then

$$\mathrm{Tor}_p^R(A, \mathrm{Tor}_q^S(B, C)) \Rightarrow \mathrm{Tor}_{p+q}^S(A \otimes_R B, C)$$

Suppose that A_R is flat. Then LHS is trivial for every $p \neq 0$, and thus

$$A \otimes_R \mathrm{Tor}_n^S(B, C) = \mathrm{Tor}_0^R(A, \mathrm{Tor}_n^S(B, C)) \cong \mathrm{Tor}_n^S(A \otimes_R B, C)$$

Similarly, define

$$F = - \otimes_R B : \mathrm{Mod}_R \to \mathrm{Mod}_S$$
$$G = - \otimes_S C : \mathrm{Mod}_S \to \mathrm{Ab}$$

then $GF = - \otimes_R (B \otimes_S C)$. Thus if $\mathrm{Tor}_i^S(Q \otimes_R B, C) = 0$ for all $i \geqslant 1$ and for all projective Q_R, we have the spectral sequence

$$\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(A, B), C) \Rightarrow \mathrm{Tor}_{p+q}^R(A, B \otimes_S C)$$

and if ${}_SC$ is flat, we have

$$\mathrm{Tor}_n^R(A, B) \otimes_S C \cong \mathrm{Tor}_n^R(A, B \otimes_S C)$$

If ${}_RB_S$ is flat on either side, at least one condition listed above is satisfied, implying

$$\mathrm{Tor}_n^S(A \otimes_R B, C) \cong \mathrm{Tor}_n^R(A, B \otimes_S C)$$

Now we consider another situation: ${}_RA$, ${}_SB_R$, ${}_SC$. Let

$$F = B \otimes_R - : {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$$
$$G = \mathrm{Hom}_S(-, C) : {}_S\mathrm{Mod} \to \mathrm{Ab}$$

The hom-tensor adjunction

$$\mathrm{Hom}_S(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

gives $GF = \mathrm{Hom}_R(-, \mathrm{Hom}_S(B, C))$. Since F is right exact covariant and G is left exact contravariant, then if $\mathrm{Ext}_S^i(B \otimes_R P, C) = 0$ for all $i \geqslant 1$ and for all projective ${}_RP$, then

$$\mathrm{Ext}_S^p(\mathrm{Tor}_q^R(B, A), C) \Rightarrow \mathrm{Ext}_R^{p+q}(A, \mathrm{Hom}_S(B, C))$$

In particular, if ${}_SC$ is injective, then

$$\mathrm{Hom}_S(\mathrm{Tor}_n^R(B, A), C) \cong \mathrm{Ext}_R^n(A, \mathrm{Hom}_S(B, C))$$

On the other hand, consider

$$F = \mathrm{Hom}_S(B,-) : {}_S\mathrm{Mod} \to {}_R\mathrm{Mod}$$
$$G = \mathrm{Hom}_R(A,-) : {}_R\mathrm{Mod} \to \mathrm{Ab}$$

Then $GF = \mathrm{Hom}_S(B \otimes_R A, -)$. Both F, G are contravariant left exact, so if $\mathrm{Ext}^i_R(A, \mathrm{Hom}_S(B,J)) = 0$ for all $i \geqslant 1$ and for all injective ${}_SJ$, we have

$$\mathrm{Ext}^p_R(A, \mathrm{Ext}^q_S(B,C)) \Rightarrow \mathrm{Ext}^{p+q}_S(B \otimes_R A, C)$$

If ${}_SB_R$ is projective on both either side, then

$$\mathrm{Ext}^n_R(A, \mathrm{Hom}_S(B,C))) \cong \mathrm{Ext}^n_S(B \otimes_R A, C)$$

and if ${}_RA$ is projective, then

$$\mathrm{Hom}_R(A, \mathrm{Ext}^n_S(B,C)) \cong \mathrm{Ext}^n_S(B \otimes_R A, C)$$

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