INTRODUCTION TO TATE THESIS

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CONTENTS

1. CLASSICAL THEORY

1.1. **Riemann zeta function.** Recall from the time being a toddler one learns the infinite sum

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The sum is absolutely convergent for $\mathrm{Re}(s) > 1$. This is the famous Riemann zeta function, first introduced and studied by Euler as a function over reals. In his 1859 article, Riemann treats $\zeta(s)$ as a function over complexes, and obtains a meromorphic continuation to the complex plane, with simple poles at $\{0, 1\}$. Moreover, he proves the functional equation:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Here $\Gamma(s)$ is the usual gamma function:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt = \int_0^\infty e^{-t}t^s\frac{dt}{t} \tag{1}$$

The integral converges absolutely for $\mathrm{Re}(s) > 0$; in fact, by a repeated use of integration by parts it is not hard to see Γ admits a meromorphic continuation to the complex plane, with simple poles at non positive integers.

1.2. **Riemann's proof.** Let's explain Riemann's proof for the functional equation. Changing variables $t \mapsto n^2 t$ in (1) for $n \geqslant 1$, we obtain

$$\Gamma(s) = \int_0^\infty e^{-n^2 t}(n^2 t)^s \frac{dt}{t} = n^{2s} \int_0^\infty e^{-n^2 t} t^s \frac{dt}{t}$$

or

$$n^{-2s}\Gamma(s) = \int_0^\infty e^{-n^2 t} t^s \frac{dt}{t}.$$

Summing over $n \geqslant 1$ and passing the sum into the integral, we get

$$\zeta(2s)\Gamma(s) = \sum_{n \geqslant 1} n^{-2s}\Gamma(s) = \int_0^\infty \left(\sum_{n \geqslant 1} e^{-n^2 t} \right) t^s \frac{dt}{t}$$

The interchange is legit as $t \mapsto \sum_{n \geqslant 1} e^{-n^2 t}$ is rapidly decreasing. As a taste of aesthetics, we do a further change of variables $t \mapsto \pi t$ and $s \mapsto \frac{s}{2}$ to get

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty \left(\sum_{n \geqslant 1} e^{-\pi n^2 t} \right) t^{\frac{s}{2}} \frac{dt}{t} \tag{2}$$

Notice RHS is defined for $\mathrm{Re}(s) > 0$, and LHS other than $\zeta(s)$ is defined for any s. In particular this gives a meromorphic continuation of ζ to $\mathrm{Re}(s) > 0$.

Now we pause and recall a famous formula in analysis:

Theorem 1.1 (Poisson summation formula)**.** For $f \in \mathcal{S}(\mathbb{R})$, one has

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)$$

Here $\widehat{f}$ denotes its Fourier transform:

$$\widehat{f}(x) := \int_{\mathbb{R}} f(y) e^{2\pi i x y} dy.$$

□

Corollary 1.1.1. For $t > 0$, one has

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t^{-1}}. \tag{3}$$

Proof. It is well known that the Fourier transform of $x \mapsto e^{-\pi x^2}$ is $e^{-\pi x^2}$. Set $f(x) := e^{-\pi x^2 t}$. Then

$$\begin{aligned} \widehat{f}(x) &= \int_{\mathbb{R}} e^{-\pi y^2 t} e^{2\pi i x y} dy \\ (y \mapsto y t^{-\frac{1}{2}}) &= \int_{\mathbb{R}} e^{-\pi y^2} e^{2\pi i x y t^{-\frac{1}{2}}} t^{-\frac{1}{2}} dy \\ &= t^{-\frac{1}{2}} e^{-\pi y^2 t^{-1}} \end{aligned}$$

Now the claimed formula follows from Poisson summation formula. □

Return to RHS of (2); we split the integral into $0 < t < 1$ and $1 < t < \infty$. Write

$$\begin{aligned}\int_0^1 \left(\sum_{n\geqslant 1} e^{-\pi n^2 t}\right) t^{\frac{s}{2}} \frac{dt}{t} &= \int_0^1 \frac{1}{2}\left(-1 + \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\right) t^{\frac{s}{2}} \frac{dt}{t} \\ &= \frac{-1}{2}\int_0^1 t^{\frac{s}{2}-1} dt + \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\right) t^{\frac{s}{2}} \frac{dt}{t} \\ \text{by (3)} &= \frac{-1}{s} + \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^{-1}}\right) t^{\frac{s}{2}-\frac{1}{2}} \frac{dt}{t} \\ (t\mapsto t^{-1}) &= \frac{-1}{s} + \frac{1}{2}\int_1^\infty \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\right) t^{\frac{1-s}{2}} \frac{dt}{t}\end{aligned}$$

Recover

$$\begin{aligned}\frac{1}{2}\int_1^\infty \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\right) t^{\frac{1-s}{2}} \frac{dt}{t} &= \frac{1}{2}\int_1^\infty t^{\frac{1-s}{2}-1} dt + \int_1^\infty \sum_{n\geqslant 1} e^{-\pi n^2 t} t^{\frac{1-s}{2}} \frac{dt}{t} \\ &= \frac{-1}{1-s} + \int_1^\infty \sum_{n\geqslant 1} e^{-\pi n^2 t} t^{\frac{1-s}{2}} \frac{dt}{t}\end{aligned}$$

For convergence issue, we must assume $0 < \mathrm{Re}(s) < 1$ so that $t^{\frac{1-s}{2}-1}$ is integrable near 0. In conclusion, for $0 < \mathrm{Re}(s) < 1$ we get the expression

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{-1}{s} + \frac{-1}{1-s} + \int_1^\infty \sum_{n\geqslant 1} e^{-\pi n^2 t} t^{\frac{1-s}{2}} \frac{dt}{t} + \int_1^\infty \sum_{n\geqslant 1} e^{-\pi n^2 t} t^{\frac{s}{2}} \frac{dt}{t} \tag{4}$$

RHS being symmetric in s and $1-s$, as a consequence we get

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

As a byproduct we obtain also a meromorphic continuation and the residues of $\zeta(s)$ from (4).

1.3. **What were important?** The first key step was to give an integral representation of $\zeta(s)$ like (2). The second was to apply Poisson summation formula to a certain infinite series.

Let us explicate more on the integral representation. Integrals of the form like (1) are exactly **Mellin transforms** which is essentially the (Fourier-)Laplace transform by a change of variables. In general, given a sequence $(a_n)_{n\geqslant 1}$, one forms a **Dirichlet series**

$$f(s) := \sum_{n\geqslant 1} \frac{a_n}{n^s}.$$

Arguing as before, we formally get

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s) = \int_0^\infty \left(\sum_{n\geqslant 1} a_n e^{-\pi n^2 t}\right) t^{\frac{s}{2}} \frac{dt}{t}.$$

In other words, Dirichlet series are nothing but a reincarnation of Mellin transforms. This has already been used intensively along with Riemann-Stieltjes integrals in analytic number theory. For a taste, see [MV07, §1.2] for example.

Poisson summation formula has also already been used in analytic number theory, especially in the point-counting problem, to change a small gap sum into a large gap sum. Readers are referred to [IK04] for this aspect.

1.4. **Generalization.** There are several generalizations to $\zeta(s)$.

(i) Dirichlet L-function. This is the Dirichlet series

$$L(s,\chi) := \sum_{n\geqslant 1} \frac{\chi(n)}{n^s}$$

where χ is a homomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \to S^1$ for some integer N.

(ii) Dedekind ζ-function. This is the series

$$\zeta_K(s) := \sum_{\mathfrak{a}\trianglelefteq\mathcal{O}_K} \frac{1}{(\#\mathcal{O}_K/\mathfrak{a})^s}$$

where K is a number field with ring of integers $\mathcal{O}_K$, and for an ideal $\mathfrak{a} \trianglelefteq \mathcal{O}_K$.

(iii) Hecke L-function.

(iv) L functions of modular forms. For a modular form f of, say, full level, this is the series

$$L(s,f) := \sum_{n\geqslant 1} \frac{a_n(f)}{n^s}$$

where $a_n(f)$ is the n-th Fourier coefficient of f at the cusp.

For (iv) we refer readers to [DS05], [Bum97], [Gel75]. In this expository article we explain how people nowadays think of (i), (ii) and (iii) via Iwasawa-Tate theory [Tat67], reinterpreted by Weil in [Wei95b] and [Wei95a]. Hecke was able to prove a functional equation for his L-function following essentially the same proof, with numbers replaced by ideal-theoretic stuff, due to the lack of unique factorization property for $\mathcal{O}_K$. See [Lan94, §13] for this account. Tate, in his thesis, found a way to get around nasty ideals, by doing harmonic analysis on a *global* space, namely ring of adeles, and on *local* spaces, namely local fields.

What was missing in the Riemann's proof is the Euler product formula:

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

interpreting the series as an infinite product of *local factors*. These are then the contributions from primes in $\mathbb{Z}$. One should view $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ in (2) as the contribution from the *infinite prime* in $\mathbb{Z}$. One usually calls $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ the **completed zeta function**, as it sorts of "closing the primes (or Spec $\mathbb{Z}$) by adding an infinite point". The local theory supplied by Iwasawa-Tate interprets local factors also as Mellin transforms. Thus the Euler product development of $\zeta(s)$ is a consequence of the fact that "global Mellin transforms factorize as local Mellin transforms".

1.5. **Organization.** We explain the local theory of Tate thesis in §2 and §3. We provide details in Archimedean setting and leave the reader to mimic the arguments for non-Archimedean setting. We include a quick introduction to p-adic numbers in §3.1 and §3.2 for those originated from analysis. The article focuses on the role played by Mellin transforms. Particularly a Paley-Wiener type theorem for Schwartz spaces are discussed §2.2 and §3.5.2. In §2 we use the language of distributions. This viewpoint is, however, suppressed in the non-Archimedean treatment in §3; see [Bum+03, §6] for this account. Finally in §4 we explain briefly the global theory.

Due to the nature of being a mixture of algebra and (harmonic) analysis, it is hard for a non number theorist to follow all the details presented here. In this article we opt to be more inclined to the analytic side. More precisely, we minimize the use of algebra. Readers only need to know the definitions of a group, ring and field. We assume, however, readers' familiarity with real analysis and general measure theory.

2. Local theory at infinity

In this section we discuss the classical Mellin transform, viewed as a *meromorphic family* of tempered distributions. To be precise, let

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |x^n f^{(m)}(x)| < \infty \text{ for all } n, m \in \mathbb{Z}_{\geqslant 0} \right\}.$$

We topologize $\mathcal{S}(\mathbb{R})$ by the seminorms $f \mapsto \sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)|$. It can be shown that $\mathcal{S}(\mathbb{R})$ is then a Fréchet space.

Definition. A **(tempered) distribution** is a continuous linear functional $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$.

People interested in the world of distributions are referred to [Hör03, §2 and §7]. However, there is no need at all to talk distributionally to obtain the result here. This is only a personal taste.

Consider the distribution $I_s : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ represented by $\mathbf{1}_{\geqslant 0}| \cdot |^{s-1}$ ($\mathrm{Re}(s) > 1$), namely

$$I_s(\phi) := \int_0^\infty \phi(t) t^{s-1} dt = \int_0^\infty \phi(t) t^s \frac{dt}{t}.$$

As said, this is the Mellin transform of ϕ. By integration by parts, for $\mathrm{Re}(s) > 0$ one has

$$I_s(\phi) = \int_0^\infty x^{s-1}\phi(x)dx = \frac{x^s \phi(x)}{s}\bigg|_0^\infty - \frac{1}{s}\int_0^\infty x^s \phi'(x) dx = -\frac{1}{s} I_{s+1}(\phi').$$

In other words,

$$I_s = \frac{1}{s}\frac{d}{dt} I_{s+1}.$$

Iterating, for $k \in \mathbb{Z}_{\geqslant 1}$ and $\mathrm{Re}(s) > -k$, one has

$$I_s = \frac{1}{s(s+1)\cdots(s+k-1)} \frac{d^k}{dt^k} I_{s+k}.$$

For $\phi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\mathrm{Res}_{s=-k} I_s(\phi) &= \lim_{s \to -k} (s+k) I_s(\phi) \\
&= \lim_{s\to -k} \frac{(s+k)(-1)^{k+1}}{s(s+1)\cdots(s+k-1)(s+k)} I_{s+k+1}(\phi^{(k+1)}) \\
&= \frac{-1}{k!} I_1(\phi^{(k+1)}) = \frac{1}{k!}\phi^{(k)}(0).
\end{aligned}$$

In other words,

$$\mathrm{Res}_{s=-k} I_s = \frac{(-1)^k}{k!}\frac{d^k}{dt^k}\delta_0$$

as distributions. Here δ_0 denotes the point mass measure at 0.

Recall the Gamma function $\Gamma(s) := I_s([t \mapsto e^{-t}])$. From the computation $\lim_{s\to -k}(s+k)\Gamma(s) = \frac{(-1)^k}{k!}$. Hence, the ratio

$$s \mapsto \mathcal{I}_s(\phi) := \frac{I_s(\phi)}{\Gamma(s)}$$

extends to an entire function with

$$\mathcal{I}_s(\phi)\big|_{s=-k} = (-1)^k \phi^{(k)}(0),$$

or

$$\mathcal{I}_{-k} = \frac{d^k}{dt^k}\delta_0$$

Let us consider the general Mellin transform on $\mathbb{R}^\times = \mathbb{R} - \{0\}$. For $\epsilon \in \{0,1\}$ and $\mathrm{Re}(s) > 1$, let

$$Z(\phi, \mathrm{sign}^\epsilon, s) := \int_{\mathbb{R}^\times} \phi(t)\mathrm{sign}(t)^\epsilon |t|^s \frac{dt}{|t|}.$$

This is the Mellin transform of $\phi \in \mathcal{S}(\mathbb{R})$ evaluated at the quasi-character $\mathrm{sign}^\epsilon |\cdot|^s$, or the distribution represented by $\mathrm{sign}^\epsilon|\cdot|^{s-1}$. One writes

$$Z(\phi, \mathrm{sign}^\epsilon, s) = I_s(\phi) + (-1)^\epsilon I_s(\phi^\vee)$$

where $\phi^\vee(t) := \phi(-t)$. Since $I_s(\phi)$ has at worst simple poles at $\mathbb{Z}_{\leqslant 0}$, it follows that so does $Z(\phi, \mathrm{sign}^\epsilon, s)$ and

$$\mathrm{Res}_{s=-k} Z(\phi, \mathrm{sign}^\epsilon, s) = \frac{1+(-1)^{\epsilon+k}}{k!}\phi^{(k)}(0).$$

Then $s \mapsto Z(\phi, \mathrm{sign}^\epsilon, s)$ has simple poles along $-\epsilon + 2\mathbb{Z}_{\leqslant 0} = -(\epsilon + 2\mathbb{Z}_{\geqslant 0})$. To kill the poles, one uses

$$L(s, \mathrm{sign}^\epsilon) := \pi^{-\frac{s+\epsilon}{2}}\Gamma(\frac{s+\epsilon}{2})$$

The factor $\pi^{-\frac{s+\epsilon}{2}}$ is added only to please number theorists. Hence the function

$$s \mapsto \frac{Z(\phi, \mathrm{sign}^\epsilon, s)}{L(s, \mathrm{sign}^\epsilon)}$$

is entire and

$$\left.\frac{Z(\phi, \mathrm{sign}^\epsilon, s)}{L(s, \mathrm{sign}^\epsilon)}\right|_{s=-(2k+\epsilon)} = \frac{(-\pi)^{-k}k!}{(2k+\epsilon)!}\phi^{(2k+\epsilon)}(0).$$

for $k \in \mathbb{Z}_{\geqslant 0}$ and $\phi \in \mathcal{S}(\mathbb{R})$.

Definition. A **quasi-character** of $\mathbb{R}^\times$ is a continuous homomorphism $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$.

Lemma 2.1. A quasi-character of $\mathbb{R}^\times$ has the form $t \mapsto \mathrm{sign}^\epsilon |t|^{s_0}$, where $\epsilon \in \{0,1\}$ and $s_0 \in \mathbb{C}$.

Proof. Refreshment. ☐

Definition. For a quasi-character $\chi = \mathrm{sign}^\epsilon |t|^{s_0}$ of $\mathbb{R}^\times$, define the **(archimedean) local L-factor**

$$L(s, \chi) := L(s + s_0, \mathrm{sign}^\epsilon).$$

Summarizing what we've obtained so far:

Theorem 2.2. For any quasi-character $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$, the map $s \mapsto \dfrac{Z(\cdot, \chi, s)}{L(s,\chi)}$ defines an entire family of tempered distributions with

$$\left.\frac{Z(\cdot, \chi, s)}{L(s, \chi)}\right|_{s=-(s_0+2k+\epsilon)} = \frac{(-1)^\epsilon(-\pi)^{-k}k!}{(2k+\epsilon)!}\delta_0^{(2k+\epsilon)}$$

where $k \in \mathbb{Z}_{\geqslant 0}$ and $\chi = \mathrm{sign}^\epsilon |\cdot|^{s_0}$ with $\epsilon \in \{0,1\}$, $s_0 \in \mathbb{C}$. Moreover, $L(s,\chi) = Z(\phi, \chi, s)$ for some $\phi \in \mathcal{S}(\mathbb{R})$. ☐

2.1. **Functional equations.**

Lemma 2.3. For $\phi, \varphi \in \mathcal{S}(\mathbb{R})$, a quasi-character χ and $0 < \mathrm{Re}(s) < 1$, one has

$$Z(\phi, \chi, s) Z(\widehat{\varphi}, \chi^{-1}, 1-s) = Z(\widehat{\phi}, \chi^{-1}, 1-s) Z(\varphi, \chi, s)$$

Proof. By Fubini, write

$$\begin{aligned} Z(\phi, \chi, s) Z(\widehat{\varphi}, \chi^{-1}, 1-s) &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \phi(x)\chi(x)|x|^s \varphi(z) e^{2\pi i yz}\chi^{-1}(y)|y|^{1-s}\frac{dx}{|x|}\frac{dy}{|y|}dz \\ (y \mapsto xy) &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \phi(x)\varphi(z) e^{2\pi i xyz}\chi^{-1}(y)|y|^{1-s} dx \frac{dy}{|y|} dz \end{aligned}$$

The last expression is symmetric in (ϕ, φ), so the lemma follows. ☐

Theorem 2.4 (Local functional equations)**.** For any quasi-character $\chi : \mathbb{R}^\times \to \mathbb{C}^\times$, there exists a unique function $\epsilon(s, \chi, \psi_\infty)$ such that

$$\frac{Z(\widehat{\phi}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi_\infty)\frac{Z(\phi, \chi, s)}{L(s, \chi)}.$$

for $\phi \in \mathcal{S}(\mathbb{R})$.

2.2. **Spectral analysis.**

Lemma 2.5. For $\phi \in \mathcal{S}(\mathbb{R})$, the Mellina transform $Z(\phi, \chi, s)$ is of rapid decay in every bounded vertical strip away the poles, namely

$$\sup_{\substack{a \leqslant \mathrm{Re}(s) \leqslant b \\ s \notin B_\varepsilon(0)+P}} (1 + |\mathrm{Im}(s)|)^N |Z(\phi, \chi, s)| < \infty$$

for any $-\infty < a \leqslant b < \infty$, $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geqslant 1}$. Here P is the set of poles of $Z(\phi, \chi, s)$ and

$$B_\varepsilon(0) + P = \{z \in \mathbb{C} \mid |z - p| < \varepsilon \text{ for some } p \in P\}.$$

If $\phi \in C_c^\infty(\mathbb{R})$, then $Z(\phi, \chi, s)$ is subexponential toward the positive infinity, namely, there exists $M > 1$ such that

$$\sup_{\substack{a \leqslant \mathrm{Re}(s) \\ s \notin B_\varepsilon(0)+P}} (1 + |\mathrm{Im}(s)|)^N M^{-\mathrm{Re}(s)} |Z(\phi, \chi, s)| < \infty$$

for any $-\infty < a$, $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geqslant 1}$.

Proof. Integration by parts. ☐

We now prove a Paley-Wiener type theorem, describes the behavior of a Schwartz function in terms of its Mellin transform. For a more general statement, see [Igu78, Theorem 1.4.2].

Define $\mathbb{H}^{\mathrm{PW}}(\mathbb{R})$ to be the space of meromorphic functions $T : \{0,1\} \times \mathbb{C} \to \mathbb{C}$ such that for any $\epsilon \in \{0,1\}$

- $s \mapsto \dfrac{T(\epsilon, s)}{L(s, \mathrm{sign}^\epsilon)}$ extends to an entire function, and
- $s \mapsto T(\epsilon, s)$ is of rapid decay on bounded vertical strips away off the poles.

The last condition is explained in Lemma 2.5. Let $\mathbb{H}^{\mathrm{PW}}_{\exp}(\mathbb{R})$ be the subspace of $\mathbb{H}^{\mathrm{PW}}(\mathbb{R})$ consisting of those T satisfying the second estimate in Lemma 2.5.

Theorem 2.6. The Mellin transform establishes an isomorphism

$$\mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{H}^{\mathrm{PW}}(\mathbb{R}).$$

which restricts to an isomorphism $C_c^\infty(\mathbb{R}) \xrightarrow{\sim} \mathbb{H}^{\mathrm{PW}}_{\exp}(\mathbb{R})$. The inverse is given by sending T to

$$x \mapsto \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{i\mathbb{R}} T(\epsilon, \sigma + r) \mathrm{sign}(x)^{-\epsilon} |x|^{-\sigma - r} \frac{dr}{2\pi i}$$

for any $\sigma \in \mathbb{R}_{>0}$.

Proof. Let $T \in \mathbb{H}^{\mathrm{PW}}(\mathbb{R})$ and define

$$\begin{aligned} f(x) &:= \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{i\mathbb{R}} T(\epsilon, \sigma + r) \mathrm{sign}(x)^{-\epsilon} |x|^{-\sigma - r} \frac{dr}{2\pi i} \\ &= \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{\sigma + i\mathbb{R}} T(\epsilon, r) \mathrm{sign}(x)^{-\epsilon} |x|^{-r} \frac{dr}{2\pi i} \end{aligned}$$

Let $k \in \mathbb{Z}_{\geqslant 0}$. By the rapid decay and the residue theorem, if we shift the contour to $-2k - \dfrac{3}{2}$ we get

$$f(x) = \frac{1}{2} \sum_{0 \leqslant \ell \leqslant 2k+1} \sum_{\epsilon \in \{0,1\}} \mathrm{sign}(x)^{-\epsilon} \mathrm{Res}_{z=-\ell} \left(T(\epsilon, z)|x|^{-z} \right) + \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{-2k-\frac{3}{2}+i\mathbb{R}} T(\epsilon, r) \mathrm{sign}(x)^{-\epsilon} |x|^{-r} \frac{dr}{2\pi i}$$

Write $T(\epsilon, z) = L(s, \mathrm{sign}^{\epsilon}) \widetilde{T}(\epsilon, z)$ for some entire function $\widetilde{T}(\epsilon, z)$. Then

$$\mathrm{Res}_{z=-\ell} \left(T(\epsilon, z)|x|^{z} \right) = \lim_{z \to -\ell} (z+\ell) L(z, \mathrm{sign}^{\epsilon}) \widetilde{T}(\epsilon, z)|x|^{-z}.$$

Recall when $\epsilon = 0$, the poles of $L(z, \mathbf{1})$ are along $2\mathbb{Z}_{\leqslant 0}$ and simple with residue

$$\mathrm{Res}_{z=-2m} L(z, \mathbf{1}) = \frac{2}{(2m)!} \frac{(-\pi)^m (2m)!}{m!} = \frac{2(-\pi)^m}{m!}.$$

Hence

$$\begin{aligned} \frac{1}{2} \sum_{0 \leqslant \ell \leqslant 2k+1} \mathrm{Res}_{z=-\ell} \left(T(0, z)|x|^{-z} \right) &= \frac{1}{2} \sum_{0 \leqslant m \leqslant k} \mathrm{Res}_{z=-2m} \left(T(0, z)|x|^{-z} \right) \\ &= \frac{1}{2} \sum_{0 \leqslant m \leqslant k} \lim_{z \to -2m} (z+2m) L(z, \mathbf{1}) \widetilde{T}(0, z)|x|^{-z} \\ &= \sum_{0 \leqslant m \leqslant k} \frac{(-\pi)^m}{m!} \widetilde{T}(0, -2m)|x|^{2m} \end{aligned}$$

When $\epsilon = 1$, the poles of $L(z, \mathrm{sign})$ are along $2\mathbb{Z}_{\leqslant 0} - 1$ and simple with residue

$$\mathrm{Res}_{z=-(2m+1)} L(z, \mathbf{1}) = \frac{2}{(2m+1)!} \frac{(-\pi)^m (2m+1)!}{m!} = \frac{2(-\pi)^m}{m!}$$

so

$$\begin{aligned} \frac{1}{2} \sum_{0 \leqslant \ell \leqslant 2k+1} \mathrm{sign}(x) \mathrm{Res}_{z=-\ell} \left(T(1, z)|x|^{-z} \right) &= \frac{1}{2} \sum_{0 \leqslant m \leqslant k} \mathrm{sign}(x) \mathrm{Res}_{z=-(2m+1)} \left(T(1, z)|x|^{-z} \right) \\ &= \frac{1}{2} \sum_{0 \leqslant m \leqslant k} \mathrm{sign}(x) \lim_{z \to -(2m+1)} (z+2m+1) L(z, \mathrm{sign}) \widetilde{T}(1, z)|x|^{-z} \\ &= \sum_{0 \leqslant m \leqslant k} \mathrm{sign}(x) \frac{(-\pi)^m}{m!} \widetilde{T}(1, -(2m+1))|x|^{2m+1}. \end{aligned}$$

Hence for each $k \in 2\mathbb{Z}_{\geqslant 0}$ we obtain

$$\begin{aligned} f(x) &= \sum_{0 \leqslant m \leqslant k} \sum_{\epsilon = 0,1} \frac{(-\pi)^m}{m!} \widetilde{T}(\epsilon, -(2m+\epsilon)) \mathrm{sign}(x)^{\epsilon} |x|^{2m+\epsilon} + \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{-2k-\frac{3}{2}+i\mathbb{R}} T(\epsilon, r) \mathrm{sign}(x)^{-\epsilon} |x|^{-r} \frac{dr}{2\pi i} \\ &= \sum_{0 \leqslant m \leqslant k} \sum_{\epsilon = 0,1} \frac{(-\pi)^m}{m!} \widetilde{T}(\epsilon, -(2m+\epsilon)) x^{2m+\epsilon} + \frac{1}{2} \sum_{\epsilon \in \{0,1\}} \int_{-2k-\frac{3}{2}+i\mathbb{R}} T(\epsilon, r) \mathrm{sign}(x)^{-\epsilon} |x|^{-r} \frac{dr}{2\pi i} \end{aligned}$$

In particular,

$$\lim_{x \to 0} \frac{d^{2k+\epsilon}}{dx^{2k+\epsilon}} f(x) = \frac{(-\pi)^m (2m+\epsilon)!}{m!} \widetilde{T}(\epsilon, -(2m+\epsilon)),$$

and hence we can extend f to a smooth function on $\mathbb{R}$. That $f \in \mathcal{S}(\mathbb{R})$ follows from integration by parts.

Finally assume $T \in \mathbb{H}^{\mathrm{PW}}_{\mathrm{exp}}(\mathbb{R})$, and let $M > 1$ be the number that satisfies the estimate

$$\sup_{\substack{a \leqslant \mathrm{Re}(s) \\ s \notin B_{\varepsilon}(0)+P \\ \epsilon \in \{0,1\}}} (1 + |\mathrm{Im}(s)|)^N M^{-\mathrm{Re}(s)} |T(\epsilon, s)| < \infty$$

for any $-\infty < a, \varepsilon > 0$ and $N \in \mathbb{Z}_{\geqslant 1}$. We claim supp $f \subseteq [-M, M]$. To see this, note for any $\sigma \geqslant 2$ that

$$\begin{aligned} |f(x)| &\leqslant \frac{1}{4\pi} \sum_{\epsilon \in \{0,1\}} \int_{\mathbb{R}} |T(\epsilon, \sigma + ir)||x|^{-\sigma} dr \\ &\ll_T \frac{1}{2\pi} \int_{\mathbb{R}} (1 + |r|)^{-2} M^{\sigma} |x|^{-\sigma} dr \\ &\ll M^{\sigma} |x|^{-\sigma}. \end{aligned}$$

If $|x| > M$, then $M^{\sigma}|x|^{-\sigma} \to 0$ as $\sigma \to +\infty$. This proves $f(x) = 0$ unless $|x| \leqslant M$ as we claim. □

Remark 2.7. A Paley-Wiener type theorem is usually referred to as a statement characterizing the image of certain function space under certain transform. Another theorem of this sort is the Schwartz-Paley-Wiener theorem [Hör03, §7.3], characterizing distributions with compact support under their Fourier transforms. One can also establishes a Paley-Wiener theorem for $C_c^{\infty}(\mathbb{R}^{\times})$ or $\mathcal{S}(\mathbb{R}^{\times})$ (functions with derivatives being rapidly decreasing toward 0 and infinity).

3. Local theory at finite places

3.1. **Absolute values.** An **absolute value** on $\mathbb{Q}$ is a map $|\cdot| : \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$ such that

(i) $|x| = 0$ if and only if $x = 0$,
(ii) $|xy| = |x||y|$,
(iii) $|x + y| \leqslant |x| + |y|$, and
(iv) $|x| \neq 1$ for some $x \in \mathbb{Q} - \{0\}$.

In particular, $|\cdot|$ restricts to a group homomorphism $\mathbb{Q}^{\times} \to \mathbb{R}_{>0}$. By (i) and (iii), $(x, y) \mapsto |x - y|$ defines a metric on $\mathbb{Q}$.

Definition. An absolute value $|\cdot|$ on $\mathbb{Q}$ is **non-Archimedean** if $|x + y| \leqslant \max\{|x|, |y|\}$ for $x, y \in \mathbb{Q}$. Otherwise it is called **Archimedean**.

Definition (Euclidean absolute value)**.** For $x \in \mathbb{Q}$, let $|x|_{\infty} := |x|$ denote the usual absolute value.

Definition (p-adic absolute value)**.** Let p be a prime. For $n \in \mathbb{Z}$, write $n = p^k n'$ with $\gcd(n, n') = 1$. Define $|n|_p := p^{-k}$. In general, for $r \in \mathbb{Q} - \{0\}$ write $r = \frac{m}{n}$ with $n \neq 0$ and let $|r|_p := |m|_p |n|_p^{-1}$.

Theorem 3.1 (Ostrowski)**.** An absolute value on $\mathbb{Q}$ is equivalent[1] either to $|\cdot|_p$ for some prime p or to $|\cdot|_{\infty}$. □

To unify the notation, we write p for either a prime or ∞. Then p will be called a **place** of $\mathbb{Q}$. When p is an actual prime, we write $p < \infty$ and say it is a **finite place**. Otherwise, $p = \infty$ and is called an **infinite place**.

Definition. For a place p of $\mathbb{Q}$, let $\mathbb{Q}_p$ denote the metric space completion of $(\mathbb{Q}, |\cdot|_p)$. For $p < \infty$, we call $\mathbb{Q}_p$ the field of p**-adic numbers**. Denote by $\mathbb{Z}_p$ the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$; this is called the ring of p**-adic integers**.

Note $\mathbb{Q}_{\infty}$ is nothing but just $\mathbb{R}$.

3.2. p**-adic numbers.** Let $p < \infty$ for the rest of the section. We have inclusions

$$\begin{tikzcd}
 & \mathbb{Q} \arrow[dr, hook, "\text{dense}"] & \\
\mathbb{Z} \arrow[ur, hook] \arrow[dr, hook, "\text{dense}"'] & & \mathbb{Q}_p \\
 & \mathbb{Z}_p \arrow[ur, hook, "\text{closed}"'] &
\end{tikzcd}$$

[1] A notion we won't define

Recall each positive integer n admits a unique p-adic representation:

$$n = a_0 + a_1 p + \cdots + a_k p^k$$

with $a_i \in \{0, \ldots, p-1\}$ and $a_k \neq 0$. One has $|n| = p^{-m}$ if and only if $a_0 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$. For negative integers, one also has a p-adic representation, but always infinite: for example,

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots$$

The sum is not convergent in euclidean topology, but is certainly convergent in $\mathbb{Q}_p$. For $n \in \mathbb{Z}$ with $\gcd(n,p) = 1$, it also admits a representation which can be obtained by reduction mod p^k. For example, take $n = 5$ and $p = 3$. Then $5^{-1} \equiv 2 \pmod 3$, $5^{-1} \equiv 2 = 2 + 0 \cdot 3 \pmod 9$, $5^{-1} \equiv 11 = 2 + 0 \cdot 3 + 1 \cdot 9 \pmod{27}$ and so on. In general, one has

$$5^{-1} \equiv a_0 + a_1 \cdot 3 + \cdots + a_k \cdot p^k \pmod{p^{k-1}}$$

and the p-adic representation is then be given by the infinite sum

$$5^{-1} = 2 + 0 \cdot 3 + 1 \cdot 9 + 2 \cdot 27 + \cdots$$

In other words, in $\mathbb{Q}_p$ every number that is coprime to p gets inverted. As an immediate consequence,

Lemma 3.2. Any rational number, and hence any p-adic number, r admits a unique p-adic representation

$$r = p^{-n}(a_0 + a_1 p + a_2 p^2 + \cdots)$$

where $n \in \mathbb{Z}$ and $a_0 \neq 0$. In addition, $|r|_p = p^n$ and $a_0 + a_1 p + a_2 p^2 + \cdots \in \mathbb{Z}_p^\times$, so that

$$\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times.$$

Here $\mathbb{Z}_p^\times$ means the group of units in the ring $\mathbb{Z}_p$.

□

Lemma 3.3. $\mathbb{Z}_p$ is a compact open subring of $\mathbb{Q}_p$. In fact,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leqslant 1\}$$
$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$$

Proof. Being an open subring is a consequence of the ultrametric inequality. The last assertion follows from the last lemma. For the compactness, let $(U_\alpha)_\alpha$ be an open cover in $\mathbb{Z}_p$. Suppose for contradiction that any finite subcollection of $(U_\alpha)_\alpha$ does not cover $\mathbb{Z}_p$. Since $\mathbb{Z}_p = \bigsqcup_{0 \leqslant a \leqslant p-1} a + p\mathbb{Z}_p$, it follows that there exists some $a_0 \in \{0, 1, \ldots, p-1\}$ such that $a_0 + p\mathbb{Z}_p$ is not covered by any finite subcollection. Iterating, we can find $a_0, \ldots, a_k \in \{0, 1, \ldots, p-1\}$ so that $a_0 + a_1 p + \cdots + a_k p^k$ is not so. Say $(a_0 + a_1 p + \cdots + a_k p^k)_k \to x$ for some $x \in \mathbb{Z}_p$. Since $x \in U_\alpha$ for some α and U_α is open, we can find $N \geqslant 0$ so that $a_0 + a_1 p + \cdots + a_k p^k \in U_\alpha$ for $k \geqslant N$. This is a contradiction, so $\mathbb{Z}_p$ is covered by some finite subcollection of $\{U_\alpha\}_\alpha$. □

Corollary 3.3.1. $\{p^n \mathbb{Z}_p \mid n \geqslant 0\}$ forms a neighborhood basis of $0 \in \mathbb{Q}_p$ consisting of compact open sets.

□

As a consequence, $\mathbb{Q}_p$ is a totally disconnected locally compact Hausdorff topological field.

3.3. **Integration on $\mathbb{Q}_p$.** Similar to the construction of the Lebesgue measure on $\mathbb{R}$, one covers a general open set in $\mathbb{Q}_p$ by the basic open sets $a + p^n\mathbb{Z}_p$. To define a measure it then suffices to assign consistently a positive number to each $a + p^n\mathbb{Z}_p$. One sets

$$\mathrm{vol}(a + p^n\mathbb{Z}_p, dx) := p^{-n}$$

for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}_p$. In particular, $\mathrm{vol}(\mathbb{Z}_p) = 1$. To see this is consistent, recall

$$\mathbb{Z}_p = \bigsqcup_{0 \leqslant a \leqslant p-1} a + p\mathbb{Z}_p.$$

Then

$$1 = \mathrm{vol}(\mathbb{Z}_p, dx) = p\,\mathrm{vol}(p\mathbb{Z}_p, dx)$$

so that $\mathrm{vol}(p\mathbb{Z}_p, dx) = p^{-1}$ is consistent. In this way we define a translation invariant measure dx on $\mathbb{Q}_p$, normalized so that $\mathrm{vol}(\mathbb{Z}_p, dx) = 1$.

For $f \in C_c(\mathbb{Q}_p)$ we can then talk about its integration

$$\int_{\mathbb{Q}_p} f(x)dx.$$

Example 3.4. Take $f(x) = \mathbf{1}_{\mathbb{Z}_p}(x)|x|_p^s$. If $\mathrm{Re}(s) > 0$, then

$$\begin{aligned}\int_{\mathbb{Q}_p} f(x)dx = \int_{\mathbb{Z}_p} |x|_p^s dx &= \sum_{n \geqslant 0} \int_{p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p} |x|_p^s dx \\ &= \sum_{n \geqslant 0} p^{-ns}\,\mathrm{vol}(p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p, dx) \\ &= \sum_{n \geqslant 0} p^{-ns}(p-1)p^{-(n+1)} = \frac{(p-1)p^{-1}}{1 - p^{-(s+1)}}\end{aligned}$$

One notices that $(p-1)p^{-1} = 1 - \frac{1}{p} = \mathrm{vol}(\mathbb{Z}_p^\times, dx)$.

We will also want to integrate over $\mathbb{Q}_p^\times$. Similar to $\mathbb{R}^\times$, we use the measure

$$d^\times x := \frac{dx}{|x|_p}.$$

In other words, for $f \in C_c(\mathbb{Q}_p^\times)$ we set

$$\int_{\mathbb{Q}_p^\times} f(x)d^\times x := \int_{\mathbb{Q}_p} f(x)|x|^{-1}dx.$$

Lemma 3.5. One has $d(ax) = |a|_p dx$ for $a \in \mathbb{Q}_p^\times$. In particular, $d^\times x$ is a multiplicatively invariant measure on $\mathbb{Q}_p^\times$.

Proof. By $d(ax) = |a|_p dx$ we actually mean

$$\mathrm{vol}(aX, dx) = |a|_p\,\mathrm{vol}(X, dx)$$

for any measurable set X of $\mathbb{Q}_p$ with finite measure. For this it suffices to show

$$\mathrm{vol}(a\mathbb{Z}_p, dx) = |a|_p.$$

Write $a = p^{-n}u$ for some $u \in \mathbb{Z}_p^\times$. Then $\mathrm{vol}(a\mathbb{Z}_p, dx) = \mathrm{vol}(p^{-n}\mathbb{Z}_p, dx) = p^n = |a|_p$ as we want. $\square$

3.4. **Fourier analysis on $\mathbb{Q}_p$.** For $x \in \mathbb{Q}_p^\times$, write

$$x = p^{-n}(a_0 + a_1 p + \cdots)$$

and set

$$\{x\}_p = p^{-n}(a_0 + a_1 p + \cdots + a_{n-1}p^{n-1}) \in \mathbb{Q}.$$

This is the **principal part** of x. Define $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ by

$$\psi_p(x) := e^{-2\pi i \{x\}_p}$$

Lemma 3.6. For $x \in \mathbb{Q}_p$, one has $\psi_p(xy) = 1$ for all $y \in \mathbb{Z}_p$ if and only if $x \in \mathbb{Z}_p$.

Proof. It amounts to showing $\{xy\}_p \in \mathbb{Z}$ for all $y \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$. This is clear. □

Definition. For $f \in L^1(\mathbb{Q}_p, dx)$, define its **Fourier transform** $\widehat{f} : \mathbb{Q}_p \to \mathbb{C}$ by

$$\widehat{f}(x) = \int_{\mathbb{Q}_p} f(y)\psi_p(xy)dy.$$

We could proceed with general integrable functions. However as in the real case it is preferable to do analysis with a convenient space where Fourier transform is an isomorphism.

Definition. The **Schwartz space** $\mathcal{S}(\mathbb{Q}_p)$ of $\mathbb{Q}_p$ is the space of locally constant functions $\mathbb{Q}_p \to \mathbb{C}$ with compact support.

In the non-Archimedean case we usually refer to local constancy as **smoothness**. Hence $\mathcal{S}(\mathbb{Q}_p)$ collects all smooth functions with compact support. One may equally writes

$$\mathcal{S}(\mathbb{Q}_p) = C_c^\infty(\mathbb{Q}_p).$$

Theorem 3.7. Fourier transform defines an isomorphism on $\mathcal{S}(\mathbb{Q}_p)$, and the Fourier inversion formula holds:

$$\widehat{\widehat{f}}(x) = f(-x).$$

One has $\widehat{\mathbf{1}_{\mathbb{Z}_p}} = \mathbf{1}_{\mathbb{Z}_p}$.

□

3.5. **Mellin transform on $\mathbb{Q}_p^\times$.**

Definition. A **quasi-character** on $\mathbb{Q}_p^\times$ is a continuous homomorphism $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

For a quasi-character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, if we write $\chi(p) = re^{2\pi i\theta}$, then

$$\chi(p^n) = r^n e^{2\pi i n\theta} = |p^n|_p^{-\log_p r} |p^n|_p^{-\frac{2\pi i\theta}{\log p}}$$

Hence for $x \in \mathbb{Q}_p^\times$ with $|x| = p^{-n}$, one has

$$\chi(x) = \chi(xp^{-n})\chi(p^n) = \chi(xp^{-n})|x|_p^s$$

for some $s \in \mathbb{C}/\frac{2\pi i}{\log p}\mathbb{Z}$. A quasi-character is then determined by a continuous homomorphism $\mathbb{Z}_p^\times \to \mathbb{C}^\times$ and a number in $\mathbb{C}/\frac{2\pi i}{\log p}\mathbb{Z}$.

Definition. For a quasi-character χ, denote by $\mathrm{wt}(\chi) \in \mathbb{R}$ the unique real number such that

$$|\chi(x)| = |x|_p^{\mathrm{wt}(\chi)}.$$

This is called the **weight** of χ.

For a measurable function $f : \mathbb{Q}_p^\times \to \mathbb{C}$, a quasi-character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ and $s \in \mathbb{C}$, we consider the Mellin transform:

$$Z(f, \chi, s) := \int_{\mathbb{Q}_p^\times} f(x)\chi(x)|x|^s d^\times x$$

whenever the integral exists.

For a Schwartz function $f \in \mathcal{S}(\mathbb{Q}_p)$, by local constancy one always has $f - f(0)\mathbf{1}_{\mathbb{Z}_p}$ is zero in a neighborhood of 0. Hence

$$Z(f, \chi, s) = Z(f - f(0)\mathbf{1}_{\mathbb{Z}_p}, \chi, s) + f(0)Z(\mathbf{1}_{\mathbb{Z}_p}, \chi, s).$$

The function $f - f(0)\mathbf{1}_{\mathbb{Z}_p}$ has compact support in $\mathbb{Q}_p^\times$, so the convergence is clear, and the first term on the right is a polynomial in $p^{\pm s}$. For the function $\mathbf{1}_{\mathbb{Z}_p}$, compute:

$$\begin{aligned} Z(\mathbf{1}_{\mathbb{Z}_p}, \chi, s) = \int_{\mathbb{Z}_p} \chi(x)|x|^s d^\times x &= \sum_{n \geqslant 0} \int_{p^n\mathbb{Z}_p - p^{n+1}\mathbb{Z}_p} \chi(x)|x|^s d^\times x \\ &= \sum_{n \geqslant 0} \int_{p^n\mathbb{Z}_p^\times} \chi(x)|x|^s d^\times x \\ &= \sum_{n \geqslant 0} \chi(p^n)|p^n|^s \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x. \end{aligned}$$

If χ is not trivial on $\mathbb{Z}_p$, then the last integral is trivial. Indeed, say $\chi(u) \neq 1$ for some $u \in \mathbb{Z}_p^\times$; then

$$\int_{\mathbb{Z}_p^\times} \chi(x) d^\times x = \int_{\mathbb{Z}_p^\times} \chi(ux) d^\times x = \chi(u) \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x$$

or

$$(1 - \chi(u)) \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x = 0.$$

Now assume χ is trivial on $\mathbb{Z}_p^\times$; then

$$Z(\mathbf{1}_{\mathbb{Z}_p}, \chi, s) = \sum_{n \geqslant 0} \chi(p^n)|p^n|^s \operatorname{vol}(\mathbb{Z}_p^\times, dx).$$

If $\mathrm{Re}(s) + \mathrm{wt}(\chi) > 0$, then

$$Z(\mathbf{1}_{\mathbb{Z}_p}, \chi, s) = \frac{\operatorname{vol}(\mathbb{Z}_p^\times, dx)}{1 - \chi(p)p^{-s}}.$$

As said, if χ is not trivial on $\mathbb{Z}_p^\times$, then

$$Z(\mathbf{1}_{\mathbb{Z}_p}, \chi, s) = 0.$$

Definition. A quasi-character χ is called **unramified** if χ is trivial on $\mathbb{Z}_p^\times$.

Definition. For a quasi-character χ, define the **local L-factor**

$$L(s, \chi) = \begin{cases} \dfrac{1}{1 - \chi(p)p^{-s}} & \text{, if } \chi \text{ is unramified} \\ 1 & \text{, otherwise} \end{cases}$$

Theorem 3.8. For a quasi-character χ and $f \in \mathcal{S}(\mathbb{Q}_p)$, the map

$$s \mapsto \frac{Z(f, \chi, s)}{L(s, \chi)}$$

extends to an entire function on the complex plane, and is a polynomial in $p^{\pm s}$. Moreover, $L(s, \chi) = Z(f, \chi, s)$ for some $f \in \mathcal{S}(\mathbb{Q}_p)$.

3.5.1. *Local functional equations.* Following exactly the same proof as in the real case, one obtains

Theorem 3.9. For any quasi-character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, there exists a unique function $\epsilon(s, \chi, \psi_p)$ such that

$$\frac{Z(\widehat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi_p)\frac{Z(f, \chi, s)}{L(s, \chi)}.$$

for $f \in \mathcal{S}(\mathbb{Q}_p)$.

□

3.5.2. *Paley-Wiener.* For a temporary use, let

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times)$$

denote the space of quasi-characters on $\mathbb{Q}_p^\times$ and

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Z}_p^\times, \mathbb{C}^\times)$$

the space of continuous homomorphisms $\mathbb{Z}_p^\times \to \mathbb{C}^\times$.

Recall there is a bijection

$$\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times) \cong \mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Z}_p^\times, \mathbb{C}^\times) \times \frac{\mathbb{C}}{\frac{2\pi i}{\log p}\mathbb{Z}}$$

so we can equip $\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times)$ with a structure of complex manifolds, with infinitely many connected components.

Define $\mathbb{H}^{\mathrm{PW}}(\mathbb{Q}_p)$ to be the space of *meromorphic* functions $T : \mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times) \to \mathbb{C}$ such that

(i) $\operatorname{supp} T$ only intersects with finitely many connected components of $\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times)$, and

(ii) $\dfrac{T(\chi, s)}{L(s, \chi)} \in \mathbb{C}[p^{\pm s}]$ for each $\chi \in \mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Z}_p^\times, \mathbb{C}^\times)$.

Again we identify $\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Q}_p^\times, \mathbb{C}^\times) \cong \mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Z}_p^\times, \mathbb{C}^\times) \times \frac{\mathbb{C}}{\frac{2\pi i}{\log p}\mathbb{Z}}$. Since $L(s, \chi) = 1$ unless χ is unramified, so $T(\chi, s) \in \mathbb{C}[p^{\pm s}]$ for all $\mathbf{1} \neq \chi$. When $\chi = \mathbf{1}$, we have $L(s, \chi) = (1 - p^{-s})^{-1}$, so $T(\chi, s)$ has at most a simple pole at $s = 0$.

Theorem 3.10. The Mellin transform $f \mapsto Z(f, \chi)$ defines an isomorphism

$$\mathcal{S}(\mathbb{Q}_p) \xrightarrow{\ \sim\ } \mathbb{H}^{\mathrm{PW}}(\mathbb{Q}_p).$$

The inverse is given by the Mellin inversion: for each $\sigma \in \mathbb{R}_{>0}$, the inverse sends T to the function

$$\mathbb{Q}_p^\times \ni x \mapsto \frac{\log p}{\mathrm{vol}(\mathbb{Z}_p^\times, dx)} \sum_{\chi \in \widehat{\mathbb{Z}_p^\times}} \int_{-\frac{\pi i}{\log p}}^{\frac{\pi i}{\log p}} T(\chi, \sigma + r)\chi^{-1}(x)|x|^{-\sigma - r}\frac{dr}{2\pi i}$$

Here we identify $T \in \mathbb{H}^{\mathrm{PW}}(\mathbb{Q}_p)$ in the last equation with a function on $\mathrm{Hom}_{\mathbf{TopGp}}(\mathbb{Z}_p^\times, \mathbb{C}^\times) \times \frac{\mathbb{C}}{\frac{2\pi i}{\log p}\mathbb{Z}}$.

□

For a proof, see [my note, §7.5.1] or [Igu78, §1.5].

4. Global theory

4.1. **Fourier analysis on Adeles.** The general idea in algebra is that fields are always easier than rings. The ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ which we will introduce soon is to $\mathbb{Q}$, pretty much like $\mathbb{R}$ is to $\mathbb{Z}$. In particular, $\mathbb{Q}$ embeds into $\mathbb{A}$ as a cocompact lattice. Anyway, $\mathbb{Q}$ is undoubtedly more flexible than $\mathbb{Z}$.

Definition. The ring of **adeles** $\mathbb{A}_{\mathbb{Q}}$ is the set

$$\mathbb{A}_{\mathbb{Q}} := \left\{ (x_p)_p \in \prod_{p \leqslant \infty} \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for all but finitely many } p < \infty \right\}$$

It is equipped with the unique (abelian) group topology so that a neighborhood basis of 0 is given by $\prod\limits_{p \leqslant \infty} U_p$, where $U_p \subseteq \mathbb{Q}_p$ is open and $U_p = \mathbb{Z}_p$ for all but finitely many $p < \infty$.

Immediately from the definition we see $\mathbb{A}_{\mathbb{Q}}$ is a locally compact Hausdorff topological ring, and $\mathbb{Q}$ embeds into $\mathbb{A}_{\mathbb{Q}}$ diagonally.

Lemma 4.1. The inclusion $\mathbb{Q} \subseteq \mathbb{A}_{\mathbb{Q}}$ has discrete image, and the quotient $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is compact.

Proof. For being discrete, it suffice to show there exists an open set U of 0 such that $U \cap \mathbb{Q} = \{0\}$. Take

$$U = (-\frac{1}{2}, \frac{1}{2}) \times \prod_{p < \infty} \mathbb{Z}_p.$$

For the compact quotient, it suffices to find a compact set K of $\mathbb{A}_{\mathbb{Q}}$ such that $K \to \mathbb{A}_{\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ is surjective. We claim

$$K = [0, 1] \times \prod_{p < \infty} \mathbb{Z}_p$$

does the job. For $(x_p)_{p \leqslant \infty}$, take $r := \sum_{p < \infty} \{x_p\}_p$; this is a finite sum as $\{y\}_p = 0$ if $y \in \mathbb{Z}_p$. Then $x_p - r \in \mathbb{Z}_p$ for all $p < \infty$. Finally take $n \in \mathbb{Z}$ so that $x_\infty - n \in [0, 1]$. One checks $(x_p)_{p \leqslant \infty} - (r + n) \in K$. $\square$

Now we define an invariant measure dx on $\mathbb{A}$. This is easy: set

$$\mathrm{vol}\left(\prod_{p \leqslant \infty} U_p, dx \right) := \prod_{p \leqslant \infty} \mathrm{vol}(U_p, dx_p)$$

where dx_p is the invariant measure on $\mathbb{Q}_p$ defined before, and $U_p \subseteq \mathbb{Q}_p$ are open sets with finite measure such that $U_p = \mathbb{Z}_p$ for all but finitely many $p < \infty$. Symbolically we write

$$dx = \prod_{p \leqslant \infty} dx_p.$$

For $f \in L^1(\mathbb{A}, dx)$ such that $f = \prod\limits_{p \leqslant \infty} f_p$ with $f_p \in L^1(\mathbb{Q}_p, dx_p)$ and $f_p = \mathbf{1}_{\mathbb{Z}_p}$ for all but finitely many $p < \infty$, we then have

$$\int_{\mathbb{A}} f(x) dx = \prod_{p \leqslant \infty} \int_{\mathbb{Q}_p} f_p(x_p) dx_p.$$

Our space of test functions, or the **Schwartz space on** $\mathbb{A}$, is defined as

$$\mathcal{S}(\mathbb{A}) := \mathrm{span}_{\mathbb{C}} \left\{ \prod_{p \leqslant \infty} f_p \mid f_p \in \mathcal{S}(\mathbb{Q}_p),\ f_p = \mathbf{1}_{\mathbb{Z}_p} \text{ for all but finitely many } p < \infty \right\} \subseteq C(\mathbb{A})$$

Define $\psi_{\mathbb{A}} : \mathbb{A} \to S^1$ by the formula

$$\psi_{\mathbb{A}}(x) := \prod_{p \leqslant \infty} \psi_p(x_p).$$

This is well-defined as ψ_p is trivial on $\mathbb{Z}_p$.

Lemma 4.2. $\psi_{\mathbb{A}}$ is trivial on $\mathbb{Q}$.

Proof. For $r \in \mathbb{Q}$, we must show

$$r - \sum_{p<\infty} \{r\}_p \in \mathbb{Z}.$$

For any $p < \infty$, note that $|r - \{r\}_p|_p \leqslant 1$ and $|\{r\}_q|_p = 1$ for $q \neq p$. Hence

$$\left| r - \sum_{q<\infty} \{r\}_q \right|_p \leqslant 1$$

for all $p < \infty$. Readers are invited to suggest themselves that this implies $r \in \mathbb{Z}$. $\square$

Corollary 4.2.1. $\psi_{\mathbb{A}}$ defines a continuous homomorphism $\psi_{\mathbb{A}} : \mathbb{A}/\mathbb{Q} \to S^1$.

Proof. We leave this to the reader as an exercise to get familiar with the topology on $\mathbb{A}$. $\square$

Definition. For $f \in L^1(\mathbb{A}, dx)$, define the **Fourier transform** $\widehat{f} : \mathbb{A} \to \mathbb{C}$ by

$$\widehat{f}(x) := \int_{\mathbb{A}} f(y)\psi_{\mathbb{A}}(xy)dy.$$

It follows that for $f = \prod_{p\leqslant\infty} f_p \in \mathcal{S}(\mathbb{A})$, one has

$$\widehat{f}(x) = \prod_{p\leqslant\infty} \int_{\mathbb{Q}_p} f_p(y_p)\psi_p(x_p y_p)dy_p = \prod_{p\leqslant\infty} \widehat{f}_p(x_p).$$

Note the product on the right is a finite product as $f_p = \mathbf{1}_{\mathbb{Z}_p}$ for all but finitely many $p < \infty$, and $\widehat{\mathbf{1}_{\mathbb{Z}_p}} = \mathbf{1}_{\mathbb{Z}_p}$.

By the Fourier theory on local pieces, one immediately gets

Lemma 4.3. Fourier transform defines an isomorphism on $\mathcal{S}(\mathbb{A})$.

$\square$

We now come to one of the promised ingredient in the proof of the functional equation:

Theorem 4.4 (Poisson summation formula)**.** For $f \in \mathcal{S}(\mathbb{A})$, one has

$$\sum_{r\in\mathbb{Q}} f(r) = \sum_{r\in\mathbb{Q}} \widehat{f}(r)$$

and the sums on both sides are absolutely convergent.

One could prove this by reducing to the classical case. In the following we explain a proof based on Fourier expansions on compact groups. For readers interested in learning the general theory of harmonic analysis and Fourier transforms on locally compact Hausdorff abelian groups, a nice place to start is [DE09].

Proof. (of Poisson summation formula) For $f \in \mathcal{S}(\mathbb{A})$, define $F : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ by

$$F(x) := \sum_{r\in\mathbb{Q}} f(r + x).$$

One checks this sum converges compactly and absolutely in $x \in \mathbb{A}$, so F is a well-defined continuous function. By Fourier theory on the compact group $\mathbb{A}/\mathbb{Q}$, one has an expansion

$$F(x) = \sum_{r\in\mathbb{Q}} \left(\int_{\mathbb{A}/\mathbb{Q}} F(y)\psi_{\mathbb{A}}(ry)dy \right) \psi_{\mathbb{A}}(rx)$$

The proof is completed by taking $x = 0$ and by the equality

$$\int_{\mathbb{A}/\mathbb{Q}} F(y)\psi_{\mathbb{A}}(ry)dy = \widehat{f}(r).$$

This is a classical trick called **unfolding**:

$$\int_{\mathbb{A}/\mathbb{Q}} F(y)\psi_{\mathbb{A}}(ry)dy = \int_{\mathbb{A}/\mathbb{Q}} \sum_{r\in\mathbb{Q}} \Big(f(s+y)\psi_{\mathbb{A}}(r(s+y)) \Big) dy = \int_{\mathbb{A}} f(y)\psi_{\mathbb{A}}(ry)dy = \widehat{f}(r).$$

□

4.2. **Mellin transforms on Ideles.** Let $\mathbb{A}^\times$ denote the group of invertible elements in the ring $\mathbb{A}$. Set theoretically,

$$\mathbb{A}^\times := \left\{ (x_p)_p \in \prod_{p\leqslant\infty} \mathbb{Q}_p^\times \mid x_p \in \mathbb{Z}_p^\times \text{ for all but finitely many } p < \infty. \right\}$$

We equip $\mathbb{A}^\times$ by the subspace topology given by the "twisted diagonal" $\mathbb{A}^\times \ni x \mapsto (x, x^{-1}) \in \mathbb{A} \times \mathbb{A}$. Equivalently, $\mathbb{A}^\times$ is topologized by the unique (abelian) group topology so that a neighborhood basis of 1 is given by $\prod_{p\leqslant\infty} U_p$, where $U_p \subseteq \mathbb{Q}_p$ is open and $U_p = \mathbb{Z}_p^\times$ for all but finitely many $p < \infty$. It is clear that $\mathbb{A}^\times$ is a locally compact Hausdorff topological abelian group.

Definition. $\mathbb{A}^\times$ is called the group of **ideles** of $\mathbb{Q}$.

The inclusion $\mathbb{A}^\times \to \mathbb{A}$ is continuous, but fails to be a topological embedding, namely, $\mathbb{A}^\times$ is not topologized using the subspace topology of $\mathbb{A}$.

Clearly $\mathbb{Q}^\times$ embeds into $\mathbb{A}^\times$ diagonally. One proves as before that the image is discrete. However, the quotient $\mathbb{A}^\times/\mathbb{Q}^\times$ fails to be compact. This is expected due to the presence of $\mathbb{R}_{>0}$.

Nevertheless, this can be fixed by restricting to the "norm 1 ideles". For $x = (x_p)_p \in \mathbb{A}^\times$, set

$$|x|_{\mathbb{A}} := \prod_{p\leqslant\infty} |x_p|_p$$

This is a finite product as $|x_p|_p = 1$ for all but finitely many $p < \infty$. Define

$$\mathbb{A}^1 := \{x \in \mathbb{A}^\times \mid |x|_{\mathbb{A}} = 1\}.$$

This is a closed subgroup of $\mathbb{A}^\times$.

Lemma 4.5. One has $\mathbb{Q}^\times \subseteq \mathbb{A}^1$, and the quotient $\mathbb{A}^1/\mathbb{Q}^\times$ is compact.

Proof. The first containment goes by the name "**Artin product formula**". For $r \in \mathbb{Q}^\times$, write $r = p_1^{n_1} \cdots p_k^{n_k}$ for its prime decomposition. Then

$$|r|_{\mathbb{A}} = |r|_\infty \prod_{i=1}^{k} |r|_p = p_1^{n_1} \cdots p_k^{n_k} \prod_{i=1}^{k} p_i^{-n_i} = 1.$$

The second results is (a special case of) a result due to Fujisaki. We must find a compact set $K \subseteq \mathbb{A}^1$ which surjects onto the quotient $\mathbb{A}^1/\mathbb{Q}^\times$. Let $x = (x_p)_p \in \mathbb{A}^1$. Let $r = \prod_{p<\infty} |x_p|_p^{-1} \in \mathbb{Q}^\times$; then $|rx_p|_p = 1$ for all $p < \infty$, so that

$$|rx|_{\mathbb{A}} = |rx_\infty|_\infty \prod_{p<\infty} |rx_p|_p = |rx_\infty|_\infty.$$

But $|r|_{\mathbb{A}} = 1$, so $|rx|_{\mathbb{A}} = |r|_{\mathbb{A}}|x|_{\mathbb{A}} = 1$ as well. Hence $|rx_\infty|_\infty = 1$. One then takes

$$K := S^1 \times \prod_{p<\infty} \mathbb{Z}_p^\times.$$

□

To define an invariant measure on $\mathbb{A}^\times$, note that $\mathrm{vol}(aX, dx) = |a|_{\mathbb{A}} \mathrm{vol}(X, dx)$ for all $a \in \mathbb{A}^\times$ and $X \subseteq \mathbb{A}$ finite measurable sets. Indeed, it suffices to show this for $X = \prod_{p \leqslant \infty} U_p$, and

$$\mathrm{vol}(aX, dx) = \prod_{p \leqslant \infty} \mathrm{vol}(a_p U_p, dx_p) = \prod_{p \leqslant \infty} |a_p|_p \mathrm{vol}(U_p, dx_p) = |a|_{\mathbb{A}} \mathrm{vol}(X, dx)$$

as claimed. Hence

$$d^\times x := \frac{dx}{|x|_{\mathbb{A}}}$$

is then an invariant measure on $\mathbb{A}^\times$.

Definition. A **Hecke character** is a continuous group homomorphism $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$.

For a Hecke character χ, one can find quasi-characters χ_p on $\mathbb{Q}_p^\times$ such that

$$\chi(x) = \prod_{p \leqslant \infty} \chi_p(x_p).$$

For this to be sensical, one just checks $\chi_p|_{\mathbb{Z}_p^\times} \equiv 1$ for all but finitely many $p < \infty$. This follows from a standard "*no small subgroup argument*". In particular, for all but finitely many $p < \infty$, the quasi-character χ_p is unramified.

Definition. For $f \in \mathcal{S}(\mathbb{A})$ and a Hecke character χ, define the Mellin transform

$$Z(f, \chi, s) := \int_{\mathbb{A}^\times} f(x)\chi(x)|x|_{\mathbb{A}}^s d^\times x.$$

whenever the integral is absolutely convergent.

Lemma 4.6. For $f = \prod_{p \leqslant \infty} f_p \in \mathcal{S}(\mathbb{A})$ and a Hecke character χ, one has

$$Z(f, \chi, s) = \prod_{p \leqslant \infty} Z(f_p, \chi_p, s)$$

and the product is absolutely convergent for $\mathrm{Re}(s)$ large.

Proof. We must compute $Z(\mathbf{1}_{\mathbb{Z}_p}, \chi_p, s)$ for χ_p unramified. But we have seen this is just $(1 - \chi_p(p)p^{-s})^{-1}$. Hence the product on the right is

$$\prod_{p \leqslant \infty} Z(f_p, \chi_p, s) = \prod_{p \in S} Z(f_p, \chi_p, s) \prod_{p \notin S} \frac{1}{1 - \chi(p)p^{-s}}$$

where S is a finite set of places such that $\infty \in S$ and χ_p is ramified for $p \in S$. It is standard that the product is convergent for $\mathrm{Re}(s) > 1$ (for example, take logarithm).

To see the equality, let $S_p = \{\infty, q \mid q \leqslant p\}$ which is a finite set of places. By monotone convergence theorem, assuming $s \in \mathbb{R}$ one has

$$Z(|f|, |\chi|, s) = \lim_{p \to \infty} \int_{\mathbb{A}^\times} \mathbf{1}_{S_p}(x)|f|(x)|\chi|(x)|x|_{\mathbb{A}}^s d^\times x$$

where $\mathbf{1}_{S_p}$ denotes the indicator of $\prod_{q \in S_p} \mathbb{Q}_q^\times \times \prod_{q \notin S_p} \mathbb{Z}_q^\times$. Take p large so that $q \notin S_p$ implies χ_q is unramified and $f_q = \mathbf{1}_{\mathbb{Z}_q}$. Then

$$\begin{aligned}
\int_{\mathbb{A}^\times} \mathbf{1}_{S_p}(x)|f|(x)|\chi|(x)|x|_{\mathbb{A}}^s d^\times x &= \int_{\mathbb{A}^\times} \left(\prod_{q \in S_p} |f_q|(x_q)|\chi_q|(x_q)|x_q|_q^s \right) d^\times x \\
&= \prod_{q \in S_p} \int_{\mathbb{Q}_q^\times} |f_q|(x_q)|\chi_q|(x_q)|x_q|_q^s d^\times x_q \\
&= \prod_{q \in S_p} Z(|f_q|, |\chi_q|, s).
\end{aligned}$$

Then

$$Z(|f|,|\chi|,s)=\lim_{p\to\infty}\prod_{q\in S_p}Z(|f_q|,|\chi_q|,s)$$

which as we've seen is finite as long as s is large. Hence $f(x)\chi(x)|x|_{\mathbb{A}}^s$ is integrable for $\mathrm{Re}(s)>1$. Replace $|f|$ by f and monotone convergence theorem by LDCT to rerun the argument. This proves the equality. □

Definition. For a Hecke character χ, define the **global L-function**:

$$L(s,\chi):=\prod_{p\leqslant\infty}L(s,\chi_p).$$

Since $L(s,\chi_p)=(1-\chi_p(p)p^{-s})^{-1}$ for all but finitely many $p<\infty$, the infinite product is absolutely convergent over some right half plane in $\mathbb{C}$. From the local theory, there exists some $f\in\mathcal{S}(\mathbb{A})$ such that

$$L(s,\chi)=Z(f,\chi,s)$$

Example 4.7. Take $f_\infty(x):=e^{-\pi x^2}$ and $f_p=\mathbf{1}_{\mathbb{Z}_p}$ and $\chi=\mathbf{1}$ the trivial character. Then

$$Z(f,\mathbf{1},s)=L(s,\mathbf{1})=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

is the completed Riemann zeta function.

Theorem 4.8 (Functional equation)**.** For $f\in\mathcal{S}(\mathbb{A})$ and a Hecke character χ, the Mellin transform

$$s\mapsto Z(f,\chi,s)$$

admits a meromorphic continuation to the complex plane, and satisfies the functional equation

$$Z(\widehat{f},\chi^{-1},1-s)=Z(f,\chi,s).$$

In addition, if we write $\chi|_{\mathbb{R}_{>0}}=|\cdot|_\infty^{s_\chi}$, then $Z(f,\chi,s)$ has at worst simple poles along $s\in\{-s_\chi,1-s_\chi\}$ with residue

$$\mathrm{Res}_{s=-s_\chi}Z(f,\chi,s)=f(0)\int_{\mathbb{A}^1/\mathbb{Q}^\times}\chi(x^1)d^\times x$$
$$\mathrm{Res}_{s=1-s_\chi}Z(f,\chi,s)=-\widehat{f}(0)\int_{\mathbb{A}^1/\mathbb{Q}^\times}\chi(x^1)d^\times x$$

Proof. For $x\in\mathbb{A}^\times$, we write $x=rx'$ for $r=|x|_{\mathbb{A}}=(|x|_{\mathbb{A}},1,1,\dots)\in\mathbb{R}_{>0}\subseteq\mathbb{Q}_\infty$ and $x^1\in\mathbb{A}^1$.

We proceed formally. By unfolding, write

$$\begin{aligned}Z(f,\chi,s)=\int_{\mathbb{A}^\times}f(x)\chi(x)|x|^sd^\times x&=\int_0^\infty\int_{\mathbb{A}^1}f(rx^1)\chi(rx^1)r^{-s}d^\times x\frac{dr}{r}\\&=\int_0^\infty\int_{\mathbb{A}^1/\mathbb{Q}^\times}\left(\sum_{a\in\mathbb{Q}^\times}f(arx^1)\right)\chi(rx^1)r^sd^\times x\frac{dr}{r}.\end{aligned}$$

Split the integral:

$$Z(f,\chi,s)=\int_0^1\int_{\mathbb{A}^1/\mathbb{Q}^\times}\left(\sum_{a\in\mathbb{Q}^\times}f(arx^1)\right)\chi(rx^1)r^sd^\times x\frac{dr}{r}+\int_1^\infty\int_{\mathbb{A}^1/\mathbb{Q}^\times}\left(\sum_{a\in\mathbb{Q}^\times}f(arx^1)\right)\chi(rx^1)r^sd^\times x\frac{dr}{r}$$

Do Poisson summation for the latter one:

$$\begin{aligned}
&\int_1^\infty \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\sum_{a\in\mathbb{Q}^\times} f(arx^1)\right) \chi(rx^1) r^s d^\times x \frac{dr}{r} \\
&= \int_1^\infty \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\sum_{a\in\mathbb{Q}} f(arx^1) - f(0)\right) \chi(rx^1) r^s d^\times x \frac{dr}{r} \\
&= \int_1^\infty \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\frac{1}{|rx^1|_{\mathbb{A}}} \sum_{a\in\mathbb{Q}} \widehat{f}(a(rx^1)^{-1}) - f(0)\right) \chi(rx^1) r^s d^\times x \frac{dr}{r} \\
(x^1 \mapsto (x^1)^{-1},\, r \mapsto r^{-1}) &= \int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\sum_{a\in\mathbb{Q}^\times} \widehat{f}(arx^1)\right) \chi^{-1}(rx^1) r^{1-s} d^\times x \frac{dr}{r} \\
&\quad - f(0) \int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi^{-1}(rx^1) r^{-s} d^\times x \frac{dr}{r} + \widehat{f}(0) \int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi^{-1}(rx^1) r^{1-s} d^\times x \frac{dr}{r}
\end{aligned}$$

Say $\chi|_{\mathbb{R}_{>0}} = |\cdot|_\infty^{s_\chi}$ for some $s_\chi \in \mathbb{C}$. Then

$$\int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi^{-1}(rx^1) r^{-s} d^\times x \frac{dr}{r} = \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi(x^1) d^\times x \times \int_0^1 r^{-s-s_\chi} d^\times r = \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi(x^1) d^\times x \times \frac{1}{-s-s_\chi}$$

and

$$\int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi^{-1}(rx^1) r^{1-s} d^\times x \frac{dr}{r} = \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi(x^1) d^\times x \times \frac{1}{1-s-s_\chi}$$

Hence

$$\begin{aligned}
Z(f,\chi,s) = &\int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\sum_{a\in\mathbb{Q}^\times} f(arx^1)\right) \chi(rx^1) r^s d^\times x \frac{dr}{r} + \int_0^1 \int_{\mathbb{A}^1/\mathbb{Q}^\times} \left(\sum_{a\in\mathbb{Q}^\times} \widehat{f}(arx^1)\right) \chi^{-1}(rx^1) r^{1-s} d^\times x \frac{dr}{r} \\
&+ \int_{\mathbb{A}^1/\mathbb{Q}^\times} \chi(x^1) d^\times x \times \left(\frac{f(0)}{s+s_\chi} + \frac{\widehat{f}(0)}{1-s-s_\chi}\right)
\end{aligned}$$

and the expression is symmetric in (f,χ,s) and $(\widehat{f},\chi^{-1},1-s)$. This finishes the proof modulo the convergence issue. We leave it to the reader. □

Corollary 4.8.1 (Functional equation for Hecke L-function)**.** For any Hecke character χ, the global L-function $L(s,\chi)$ admits a meromorphic continuation to the complex plane and there exists an entire function $\varepsilon(s,\chi,\psi_{\mathbb{A}})$ such that the functional equation holds:

$$L(1-s,\chi^{-1}) = \varepsilon(s,\chi,\psi_{\mathbb{A}}) L(s,\chi).$$

In addition,

$$\varepsilon(s,\chi,\psi_{\mathbb{A}}) = \prod_{p\leqslant\infty} \varepsilon(s,\chi_p,\psi_p).$$

and $\varepsilon(s,\chi_p,\psi_p) = 1$ for all but finitely many $p < \infty$.

□

One can specify the poles of $L(s,\chi)$ and the residues. We again leave it to the reader.

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